Quality Ladders, Competition and Endogenous Growth

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Abstract: We examine a competitive theory in which new ideas are introduced only when diminishing returns to the use of existing ideas sets in. After an idea is introduced, the capital associated with that idea expands, and the price of the idea falls. Once the price falls far enough, it becomes profitable to introduce a new, costlier, idea. The resulting competitive theory is consistent with fixed costs of innovation, no more difficult than the existing theory of monopolistic innovation, and accounts for the same basic facts. However, there is evidence that innovation is driven by diminishing returns on existing ideas – a fact that the existing theory does not account for.

1. Introduction

The standard view of innovation is that economic progress climbs a quality ladder, driven by the incentive of short-term monopoly power. In this traditional view the presence of fixed costs make short-term monopoly power essential to innovation. This traditional view has been the primary theoretical tool in accounting for the dependence of technological progress on economic fundamentals such as patience and cost. Important examples of this line of research are the models of Romer [1990], Grossman and Helpman [1991], and Aghion and Howitt [1992]. Although these models are based on increasing returns to scale, the speed of innovation is limited by diminishing returns. Innovation is unambiguously good, but as the rate of innovation increases, the marginal cost of innovation goes up, limiting the rate both in equilibrium and at the social optimum.

Our own examination of innovation suggests a rather different perspective, and an alternative story. Each innovation opens the door to growth on a new rung of the quality ladder: knowledge and physical capitals are accumulated that embody and exploit the new idea. As the investment opportunities opened by an innovation are exhausted, it becomes both socially and privately optimal to introduce a new one. In this process fixed costs and monopoly power play no essential role.

Examples that illustrate the fundamental difference between the two theories abound, so we should not dwell with many of them and just consider a paradigmatic few. The existing theory predicts that after radio was invented the inventors would move at once to inventing television. Our theory predicts that they would continue to spend their resources improving and expanding the production of radios. Only after the radio became widespread, and the gains to further improvement and expansion of radio technology became small, would they move to work on television. Our prediction is of course an accurate description of the actual economic facts in the invention of radio and television – and the history of R&D shows it is the rule, not the exception. After an invention, successful inventors do not turn immediately to inventing something else, but rather to promoting and improving their existing invention. After inventing the light-bulb, Thomas Edison turned primarily to investing in and promoting electrical power and selling light-bulbs – not to inventing the fluorescent light or the LED. Other inventors did not turn to the fluorescent light right away either: this did not become profitable until the Edison's light-bulb had been throughoutly developed and exploited. And every movie producer can tell you, the time to release your great new blockbuster adventure movie is not two days after your rival has done the same.

Our goal in this paper is to give an account of the benchmark competitive theory of innovation, in the presence of fixed costs, that captures the idea that innovation opens opportunities, and that additional innovation is not desirable until those opportunities have been exploited through capital accumulation.

2. The Grossman-Helpman Model

There are a variety of models of quality ladders with fixed costs, increasing returns, and external effects, most notably those of Romer [1990], Grossman and Helpman [1991], and Aghion and Howitt [1992]. We adopt the model of Grossman and Helpman [1991] as a particularly clean example that leads to a simple closed form solution and includes a straightforward welfare analysis. Here we summarize their results, employing their notation throughout.

Goods come in different qualities.[1] Denote by d_j the consumption (demand) for goods of quality j, let ρ be the subjective interest rate, and let $\lambda > 1$ be a constant measuring the increase in quality as we move one step up the quality ladder. We let

$$c_t = \sum_j \lambda^j d_{jt}$$

denote quality adjusted aggregate consumption. Utility of the representative consumer is

$$U = \int_0^\infty e^{-\rho t} \log[\, c_t \,] dt \,.$$

One unit of output of each quality requires just a unit of labor to obtain. The first firm to reach step j on the quality ladder is awarded a legal monopoly over that technology. This monopoly lasts only until there is a new innovation and technology $j+1$ is introduced, at which time all firms have access to technology j. This is the same device used by Aghion and Howitt, and has an obvious convenience for solving the model. Taking labor to be the numeraire, the implication is that the price of output of

[1] In the original Grossman-Helpman paper there were a continuum of identical sectors indexed by ω. Since this plays little role in the analysis, and for notational simplicity, we omit it here.

technology $j+1$ relative to that of technology j is given by the limit pricing formula $p=\lambda$.

The intensity of R&D for a firm is denoted by $\tilde{\iota}$, and the probability of successfully achieving the next step during a period of length dt is $\tilde{\iota}dt$ at a cost of $\tilde{\iota}a_I dt$.

Let E denote the steady state flow of consumer spending. Since the wage rate is numeraire and price is λ the monopolist gets a margin of $\lambda-1$ on each unit sold. His share of expenditures is therefore his margin divided by the price, that is $(\lambda-1)/\lambda = 1-1/\lambda$ of E. Since the cost of getting the monopoly is a_I, the rate of return is $(1-1/\lambda)E/a_I$. However, there is a chance ι of losing the monopoly, reducing the rate of return by this amount. Since in steady state expenditure is constant, the interest rate in expenditure units is equal to the subjective interest rate. Equating the rate of return to the subjective interest rate gives the Grossman and Helpman equation determining research intensity

$$\frac{(1-1/\lambda)E}{a_I} - \iota = \rho .$$

There is a single unit of labor, the demand for which comes from the $a_I\iota$ units used in R&D and the E/λ units used to produce output.[2] Consequently, the resource constraint is

$$a_I\iota + E/\lambda = 1 .$$

Notice that the same labor is used for R&D as is used to produce output. This captures the sensible idea that there is increasing marginal cost of R&D. That the increasing cost is due to resources being sucked out of the output sector is analytically convenient. It implies that the cost of R&D, measured in units of output, is proportional to the current rung on the quality ladder, making possible steady state analysis.

These two equations can be solved for the steady state research intensity

$$\iota = \frac{(1-1/\lambda)}{a_I} - \frac{\rho}{\lambda} .$$

By contrast the social optimum research intensity is derived by calculating steady state utility to be $[\log E - \log\lambda + (\iota/\rho)\log\lambda]/\rho$. Since the optimal plan in a steady

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[2] We have simplified Grossman and Helpman here by normalizing the stock of fixed labor to one.

state maximizes the steady state utility subject to the resource constraint, simple algebra gives the optimum

$$\iota^* = \frac{1}{a_I} - \frac{\rho}{\log \lambda}.$$

3. Climbing the Ladder under Competition

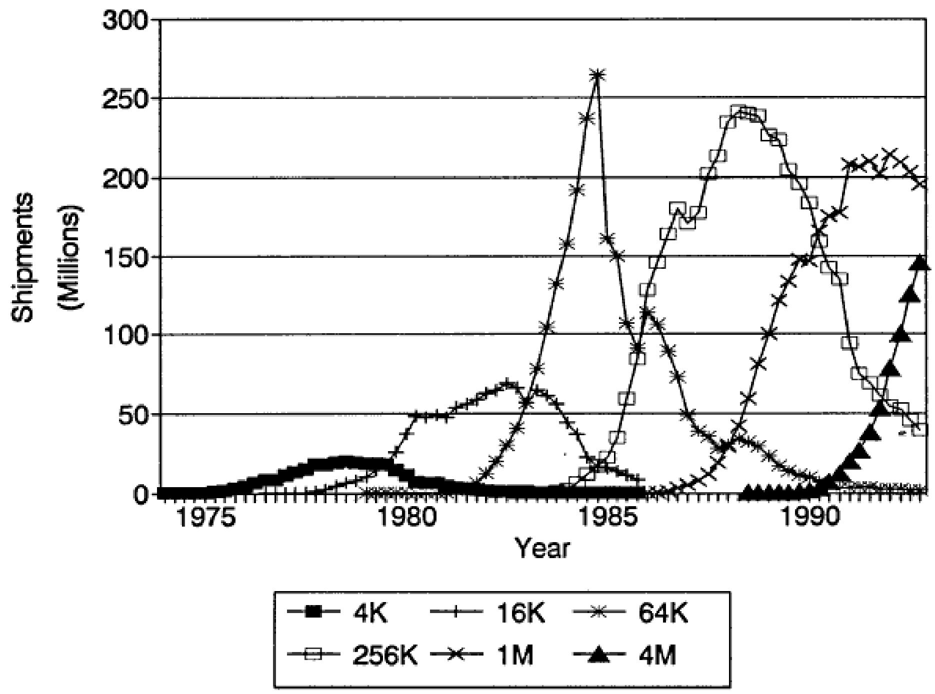
The account given by Grossman and Helpman is one in which fixed costs and monopoly power play a key role. In particular, if the fixed cost of research intensity a_I goes to zero, both the equilibrium and the optimal research intensities go to infinity. Conversely, if monopoly power goes to zero so does research intensity, and economic growth along with it.

Diminishing returns also play a role in the model. As R&D increases, it becomes more costly relative to output, while its benefits do not increase correspondingly. These diminishing returns determine the equilibrium and optimal research intensities, while it is monopoly power that, by overcoming the obstacle imposed by the fixed cost, makes such intensity positive, hence innovation and growth possible.

No doubt, diminishing returns in the R&D sector are part of the reason that we do not move up the quality ladder more quickly. But is it the only reason, or even the most important reason? Consider the example from the introduction of radio and television. Suppose that the technology of radio has just been invented. Do we expect that if there were constant returns in the R&D sector the market would be instantly flooded with radios and then an immediate effort put forward to invent TVs? Or is it more reasonable to suppose that initially only a few radios are produced and that their price is quite high since there are only a few of them and it is costly to increase their production quickly? Because the market value of radios is high, R&D resources are allocated to expanding their production and not to inventing television right away. In this story, as radio production ramps up, the price of radios diminishes and eventually becomes so low that it makes sense to invest resources to inventing and producing TVs.

The story of endogenous innovation on a quality ladder we want to tell is one where the incentives for innovating come about because of diminishing returns to making use of existing inventions. Because inventing a new good always costs more than using an existing one, it becomes profitable to invent the new good only when the quantity of the old good around is large enough to make its price low relative to that of the new one.

Before presenting a model of endogenous innovation due to diminishing returns, it is useful to look at a typical example of how real quality ladders work, borrowed from Irwin and Klenow [1994]. The good in question is the DRAM memory chip. The different qualities correspond to the capacity of a single chip. The figure below, showing shipments of different quality chips, is reproduced from that paper. The key fact is that production of a particular quality does not jump up instantaneously but ramps up gradually, and that a new quality is introduced when the stock of the old one is fairly



large. Further, the old vintage is phased out gradually as the new vintage is introduced. Their price data shows that the price of each vintage of chip falls roughly exponentially over the product cycle – meaning the incentive to introduce the next generation chip keeps increasing. This vividly portrays our story for the semiconductors industry. Overwhelming evidence suggests this is the usual pattern in most industries.[3] The question this evidence poses, and the intuition upon which our model of endogenous

[3] See, for example, Hannan and McDowell [1987], Manuelli and Seshadri [2003], Rose and Joskow [1990], Sarkar [1998]

growth with quality ladders is built upon, is: why introduce a new product if the old one is still doing so well?

4. Innovation with Knowledge Capital

We adopt the same demand structure as Grossman and Helpman, that is quality adjusted consumption is

$$c_t = \sum_j \lambda^j d_{jt}$$

and the preferences for the representative consumer are

$$U = \int_0^\infty e^{-\rho t} \log[c_t] dt .$$

However, we assume that output is produced both from labor and existing specialized productive capacity. For simplicity we identify “productive capacity” with knowledge and assume that different rungs on the quality ladder correspond to different qualities of capital and knowledge used to produce that particular output. We denote by k_j the combined stock of capital and knowledge that goes into producing quality j output. By explicitly modeling the stock of knowledge, we can distinguish between investment on a given rung – spreading and adopting knowledge of a given type through teaching, learning, imitation, and copying – and investment that moves between rungs – innovation or the creation of new knowledge. We refer to k_j as *quality j knowledge capital*. In practice knowledge capital can have many forms – it can be in the form of human knowledge or human capital, but it can also be embodied in physical form, such as books, or factories and machines of a certain design.

Knowledge capital has two uses: it can be used either to generate more knowledge capital or to produce consumption. More knowledge capital means either increasing the stock of the same quality of knowledge capital or creating a higher quality. If quality j knowledge capital is used to produce more knowledge capital of the same quality, it does so at a fixed rate $b > \rho$ per unit of knowledge capital input. In other words, on any given rung of the quality ladder, the production function is linear in knowledge capital used as input. More radios can be produced, as the existing technology for producing radios is imitated and additional knowledge capital of that kind accumulated. This can be regarded as capital widening or competitive imitation.[4]

We also allow for innovation – that is, the production of a higher quality of knowledge capital from an existing quality. This can be regarded as capital deepening. Specifically, a unit of knowledge capital of quality j can be produced from $a > 1$ units of quality $j - 1$. This represents the conversion of knowledge capital from, say, that capable of producing radios to one that can produce TVs. In the strict interpretation of k_j as a stock of knowledge about how to produce, capital widening corresponds to the spread of existing knowledge, and capital deepening to the creation of new knowledge from old.[5]

Alternatively, knowledge capital of quality j can be employed in the production of quality j consumption on a one-to-one basis. As in the Grossman and Helpman model, output of consumption also requires labor – leading to diminishing returns for each quality of knowledge capital. Specifically, each unit of quality j knowledge capital employed in the production of consumption requires a single unit of labor, and this produces a unit flow of quality j consumption. As before, we normalize the fixed labor supply to one. Our retained assumption is that, when measured in units of current consumption, creation of new knowledge is costlier than spreading knowledge already in existence, that is $b > \lambda / a.$. This implies that, as long as it is not needed for expanding consumption – that is, until all labor becomes employed with the most advanced kind of capital – it is not socially efficient to introduce a new kind of knowledge capital in the production of consumption.

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[4] As in Grossman-Helpman, the assumption of linearity is purely for algebraic convenience: any sufficiently productive concave function would also do.

[5] Notice that new capital loses the capability of the old capital that was converted. This may be true for the physical replacement of machines with newer models, but is not usually the case for human capital – a pilot may still be capable of flying a Cessna after learning to fly a Stealth bomber. However, the key constraint is that at any moment in time you must be acting as one of the two kinds of pilot, but not as both. If we introduced a technology for converting quality $j + 1$ knowledge capital back to quality j at the same ratio as the forward conversion, then this would precisely capture knowledge that was not lost, but knowledge capital as a resource that could be deployed at only one level at a time. However in the divisible case there is no reason to use the backward conversion technology, so the equilibrium would not change.

Let h_j denote the flow investment of knowledge capital of quality j in the production of knowledge capital of quality $j+1$. Under our assumptions the motion of quality j stock of knowledge capital is then given by

$$\dot{k}_j + h_j = b(k_j - d_j) + \frac{h_{j-1}}{a}.$$

Notice that d_j of the stock of capital must be allocated to the consumption sector and so the amount available for conversion into new knowledge capital of any quality is $k_j - d_j$. We require that $d_j \le k_j$ and $h_j \ge 0$. The flow of new knowledge capital must be divided into a variation in existing knowledge capital $\dot{k}_j$ and investment in higher quality knowledge capital h_j. Note that over a short period of time, there is no limit (other than $k_j - d_j$) on the size of the flow from quality j capital to quality $j+1$; that is h_j may be arbitrarily large, provided $\dot{k}_j$ is correspondingly negative, so we allow also discrete conversion $\Delta k_{j+1} = -\Delta k_j / a$.

The key technical fact is that this economy is an ordinary diminishing return economy: both the first and second welfare theorems hold, so efficient allocations can be decentralized as a competitive equilibrium and vice versa.[6]

5. Competitive Equilibrium with Knowledge Capital

Our first goal is to characterize the competitive equilibrium of the knowledge capital model. We will examine the robustness of the equilibrium to the assumptions of the model later. First, we will show that - after a possible *initial unemployment phase* when there is too little capital to employ the entire stock of labor - as time goes by the competitive equilibrium settles into a steady state cycle along which the aggregate stock of productive capacity grows at an oscillating rate. The cycle alternates between a *growth* phase in which consumption grows at the rate $b - \rho$ as the stock of capital used to produce it is upgraded from quality j to quality $j+1$, and a *build-up* phase in which consumption remains flat while knowledge capital of the most recent kind is accumulated. The growth phase ends when it is no longer possible to increase

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[6] The welfare theorems are proven for the discrete time version of this model by Boldrin and Levine [2001]. Note that in that paper we assumed that one unit of capital of type j and λ^{-j} units of labor were required to produce a unit of quality j consumption, and that the capital is used up in the process of producing consumption. This simply changes the units in which capital is measured. The assumptions here are chosen for compatibility with Grossman and Helpman [1991].

consumption without innovation. During the build up phase knowledge capital is accumulated and its price drops; eventually it becomes cheap enough to use in the production of consumption, and the next growth phase begins. Research intensity, the inverse of the combined length of the two phases is

$$j^* = \frac{b - \rho}{\log a}.$$

The Pricing of Knowledge Capital

The key to understanding the competitive equilibrium is the pricing of knowledge capital. We take current utility as numeraire implying that the current price of consumption is marginal utility that in our logarithmic case is equal to $1/c_t$. Define q_{jt} to be the time t price of quality j knowledge capital. We first examine the value of knowledge capital in the production of more knowledge capital.

Knowledge capital can be used for innovation – that is to create higher quality knowledge capital. Zero profits on innovation implies that $q_{j+1,t} - aq_{jt} \leq 0$, or equivalently, $q_{j,t+1} / q_{jt} \leq a$, with equality holding when innovation is taking place. Knowledge capital can also be used for imitation – that is to create more knowledge capital of the same quality. In this case the rate of return is the growth rate b, so we must have this return plus capital gains equal to the subjective interest rate, that is, $b + \dot{q}_{jt} / q_{jt} = \rho$, or equivalently $\dot{q}_{jt} / q_{jt} = -(b - \rho)$.[7]

The fact that the price of knowledge capital is necessarily falling over time is significant. Consider, for example, the fact that the first mover must incur greater costs than subsequent competitors. This is true here, since the first innovator to produce quality $j+1$ from quality j must pay more for knowledge capital as an input. However: by virtue of being first, he can also sell his freshly created knowledge for a higher price than his imitators, who must sell at a later date when the new knowledge is worth less. In other words, even in this model of perfect competition, there is a first mover advantage, because the competitive price of output is falling over time.

[7] Strictly speaking, these conditions need only hold with inequality if knowledge capital is not being used to produce more knowledge capital. For example, if there is no knowledge capital of quality J or higher, then we can have $q_{j+1,t} < aq_{jt}$ for $j+1 \geq J$. However, there exist equivalent equilibria in which profits are zero. See Boldrin and Levine [2001].

Consumption Value of Knowledge Capital: Initial Unemployment Phase

Next we turn to the value of knowledge capital used to produce consumption. When the economy begins, the initial condition may be such that there is insufficient capital to employ all the labor. We first examine the value of knowledge capital used to produce consumption during this phase. The key idea is that the relative value of quality j and $j+1$ knowledge capital used in producing consumption is λ, while the price ratio we know must be $a > \lambda$. This implies that only the lowest quality of knowledge capital is used to produce consumption. Subsequently we will establish that when there is full employment, no more than two qualities of knowledge capital are used to produce consumption.

Let us examine carefully the value of knowledge capital used to produce consumption when there is unemployment. Start from the observation that there are two different ways we can use a small amount ε of k_j over some short time period τ. We can produce either consumption or more capital. If additional consumption is to be produced, we move ε units into the consumption sector resulting in $\lambda^j \varepsilon\tau$ units of consumption. On the other hand, if εk_j is used to produce more capital of the same quality, we get $b\varepsilon\tau$ new units. Hence one unit of k_j is a perfect substitute for λ^j / b units of consumption. Since the marginal social value of c_t units of quality adjusted consumption is $1/c_t$, the marginal social value of a unit k_j in producing output is $q_{jt}^U \equiv \lambda^j / bc_t$. When there is unemployment, the price of k_j cannot be less than this: we must have $q_{jt} \geq q_{jt}^U$, with equality if k_j is actually used to produce consumption. As claimed, when there is unemployment if quality j' knowledge capital is used to produce consumption, then no higher quality can be used. That is if quality $j > j'$ was used to produce consumption, then we would have $q_{jt} / q_{j't} = q_{jt}^U / q_{j't}^U = \lambda^{j-j'}$. This would, however, contradict the zero profit on innovation condition that $q_{jt} / q_{j't} = a^{j-j'}$.

Consequently, during this *initial unemployment phase* a single quality j of capital is used to produce consumption. The price of this capital is $q_{jt} = q_{jt}^U = \lambda^j / bc_t$. Since $\dot{q}_{jt} / q_{jt} = -(b-\rho)$ from the no profit on imitation condition, it must be that consumption grows at the constant rate $\dot{c}_t / c_t = b - \rho$. Eventually, then, consumption grows sufficiently large that $c_t = \lambda^j$, meaning that we have reached full employment and that it is no longer possible to increase output by employing more quality j capital.

Consumption Value of Knowledge Capital: Full Employment

We next examine the value of knowledge capital used to produce consumption when there is full employment. Suppose that a positive amount of quality $j' < j$ capital is used to produce consumption. In this case additional units of k_j can be used to increase consumption by replacing the same number of units $k_{j'}$, which are of inferior quality. Notice that the lowest quality capital used in producing consumption has no marginal value in consumption: additional units cannot be used to increase consumption at all.

As is the case during the initial unemployment phase, there are two different ways that we can use a small amount ε of k_j over some short time period τ. We can produce more consumption or more knowledge capital. The key difference is that now when we move εk_j into the consumption sector, we must displace existing knowledge capital of some other quality to free up the labor needed to work with εk_j. This means that when we move εk_j into the consumption sector, we necessarily free a similar amount of j', and of course this capital has social value. The easiest way to do the computation is to imagine that the newly freed $\varepsilon k_{j'}$ is converted immediately to quality j. Consequently, increasing of a unit the quality j productive capacity of consumption, requires fewer units than this as input. Specifically, if we increase quality j capital used to produce consumption by $\varepsilon/(1-1/a^{j-j'})$ units, we displace the same amount of quality j' capital. These $\varepsilon/(1-1/a^{j-j'})$ units of $k_{j'}$ are converted into $1/a^{j-j'}$ times as many units, that is, $\varepsilon/(a^{j-j'}-1)$ units of k_j. This means that the net usage of k_j in this whole operation is only

$$\frac{\varepsilon}{1-1/a^{j-j'}} - \frac{\varepsilon}{a^{j-j'}-1} = \varepsilon\,.$$

In other words, if we move ε units of quality j capital to displace quality j' capital in the production of consumption, the amount of quality j consumption productive capacity increases by the greater amount $\varepsilon/(1-1/a^{j-j'})$.

From this, we conclude that using an additional ε units of k_j to displace ε units of the inferior quality $k_{j'}$ in the production of consumption increases the output of the latter by

$$\frac{\lambda^j - \lambda^{j'}}{1-1/a^{j-j'}}\varepsilon\tau\,,$$

where the numerator is the productivity differential between the high and low quality capital.

As was the case with unemployed labor, if during the same small interval of time τ, ε units of quality j capital are used to produce more of itself, we get $b\varepsilon\tau$ new units. Hence, in the full employment case when quality j' capital is the lowest quality still used to produce consumption, we conclude that one unit of k_j is a perfect substitute for

$$\frac{\lambda^j - \lambda^{j'}}{b(1 - 1/a^{j-j'})}$$

units of consumption. This gives the marginal social value of a unit of k_j added to the production of consumption by displacing quality j' knowledge capital as

$$q_{jt}^{j'} = \frac{\lambda^j - \lambda^{j'}}{b(1 - 1/a^{j-j'})}\frac{1}{c_t}.$$

The price q_{jt} of quality j knowledge capital cannot be lower than this, since it cannot be strictly profitable to buy knowledge capital and shift it into the production of consumption, so

$$q_{jt} \geq q_{jt}^{j'},$$

with equality if the knowledge capital is actually used to produce consumption.

There is one more important conclusion we can draw from this analysis of the consumption value of knowledge capital – only two qualities of knowledge capital are actually used to produce consumption. Consider the following inequality:

$$\begin{aligned} q_{jt}^{j'} &= \frac{\lambda^j - \lambda^{j'}}{b(1 - 1/a^{j-j'})}\frac{1}{c_t} = a^j\left(\frac{\lambda}{a}\right)^{j'}\frac{\lambda^{j-j'} - 1}{a^{j-j'} - 1}\frac{1}{bc_t} \\ &= a^j\left(\frac{\lambda}{a}\right)^{j'}\frac{\lambda^{j-j'-1} + \lambda^{j-j'-2} \ldots + 1}{a^{j-j'-1} + a^{j-j'-2} \ldots + 1}\frac{\lambda - 1}{a - 1}\frac{1}{bc_t}. \\ &< a^{j-j'-1}q_{j'+1,t}^{j'} \end{aligned}$$

Suppose then that, when quality j' is, quality $j > j'+1$ is also used to produce consumption. We have $a^{j-j'-1}q_{j+1,t} = q_{jt} = q_{jt}^{j'} < a^{j-j'-1}q_{j+1,t}^{j'}$, which means that it is strictly profitable to use quality $j'+1$ in consumption – an impossibility. We conclude that, when there is full employment, at most two adjacent qualities of knowledge capital $j-1, j$ are used to produce consumption. When this is the case, we must have for quality j price $q_{jt} = q_{jt}^{j-1}$. In particular, $\dot{q}_{jt}^{j-1} / q_{jt}^{j-1} = \dot{q}_{jt} / q_{jt} = -(b-\rho)$, implying once again that consumption grows at $b-\rho$.

The Growth Cycle

We are now in a position to analyze what happens at the end of the initial phase when full employment is reached. We may assume without loss of generality that the initial quality of knowledge used is quality $j-1=0$, so that when a single unit is employed in producing consumption, we have $c_t = 1$. The price $q_{0t} = q_{0t}^U = 1/b$, from which we can see that the price of quality $j=1$ knowledge capital is $q_{1t} = a/b$. The value of this knowledge capital in producing consumption is

$$q_{1t}^0 = \frac{\lambda - 1}{b(1-1/a)} = \frac{a}{b}\frac{\lambda-1}{a-1} < q_{1t}.$$

That is, it does not at this time pay to use quality $j=1$ capital to produce consumption. Because this is now also the only way to increase its output, it follows that consumption must remain constant; as long as output remains constant, so does the value of capital in producing consumption $q_{1,t+\tau}^0 = q_{1t}^0$. On the other hand, because it is being accumulated, the price of quality j capital is falling, $q_{j,t+\tau} = (a/b)e^{-(b-\rho)\tau}$. When

$$q_{1t}^0 = \frac{a}{b}\frac{\lambda-1}{a-1} = (a/b)e^{-(b-\rho)\tau} = q_{1t}$$

that is at

$$\tau = \frac{1}{b-\rho}\log\frac{a-1}{\lambda-1}$$

it becomes profitable to introduce quality j knowledge capital into producing consumption, at which point consumption resumes growth at the rate $b-\rho$.

We refer to the period during which consumption is growing as the *growth phase*. In general, when quality j capital is first introduced, quality adjusted consumption is λ^{j-1}, while further increases in consumption require quality $j+1$ as soon as consumption reaches λ^j. During the growth phase, qualities $j-1$ and j are used to produce consumption, and consumption grows at $b-\rho$. Hence the length of the growth phase τ^g is characterized by

$$\lambda^{j-1}e^{(b-\rho)\tau^g} = \lambda^j$$

meaning that the length of the growth phase is

$$\tau^g = \frac{\log \lambda}{b - \rho}.$$

At the end of the j-th growth phase, because of zero-profit in its production, the price of quality $j+1$ capital must satisfy

$$q_{j+1,t} = aq_{jt} = aq_{jt}^{j-1} = a\frac{\lambda^j - \lambda^{j-1}}{b(1-1/a)}\frac{1}{\lambda^j} = \frac{\lambda - 1}{b(1-1/a)}\frac{a}{\lambda}.$$

At the same time the consumption value of quality $j+1$ capital is

$$q_{j+1,t}^j = \frac{\lambda^{j+1} - \lambda^j}{b(1-1/a)}\frac{1}{\lambda^j} = \frac{\lambda - 1}{b(1-1/a)} < q_{j+1,t}$$

In other words, at the end of the growth phase, we must have a *build-up* phase, during which consumption remains fixed at λ^j, while the price of quality $j+1$ capital falls by a factor of λ/a. Since it falls at the constant rate $b-\rho$, this takes

$$\tau^b = \frac{\log a - \log \lambda}{b - \rho}.$$

Following this, we again begin the growth phase, and the cycle repeats at the next level of the quality ladder.

The intensity of innovation is the rate at which we move up the knowledge capital ladder. This is just the inverse of the length of the cycle, that is of the sum $\tau^g + \tau^b$ of the two parts

$$j^* = \frac{b - \rho}{\log a},$$

as promised.

6. Comparison of the Models

We have, now, three possible models explaining endogenous growth. One is the Grossman-Helpman model, in which the innovation rate is given by

$$\iota = \frac{(1 - 1/\lambda)}{a_I} - \frac{\rho}{\lambda}.$$

Another is the efficient solution of the Grossman-Helpman model – which, as we will argue below, may correspond better to real institutions than the particular model of monopolistic competition they propose. Here the innovation rate is given by

$$\iota^* = \frac{1}{a_I} - \frac{\rho}{\log \lambda}.$$

Finally, we have the model of competitive knowledge-capital accumulation, in which the innovation rate is given by

$$\jmath^* = \frac{b - \rho}{\log a}.$$

Qualitatively, in each case the innovation rate is a similar function of the cost of innovating and the degree of impatience. As consumers are more patient, the frequency of innovation goes up. As it becomes more costly to innovate, innovation goes down. There are of course minor differences in the functional forms between these solutions. But the functional forms depend on a variety of assumptions – log utility, exponentially improving steps, and so forth, that were contrived to obtain a closed form solution, so the particular functional forms have no strong claim of correctness. Moreover, the models differ in ways that are also designed to ease the solution. For example, it is technically convenient for Grossman and Helpman to assume the same labor is used in producing knowledge as in producing output. However, it is technically convenient for us to assume that knowledge capital is produced only from knowledge capital.

What are then the substantive, as opposed to the technically convenient, differences? First, the parameter λ, how high each rung of the ladder is, has no effect in our model – this is due to the presence of two offsetting effects, increasing the intensity of innovation during the first build-up phase of the cycle – this effect being present also in Grossman-Helpman – and decreasing it during the second growth phase. However, this "neutrality" of step size in our model is due to the use of linear technologies, and it is likely that dropping such simplifications could yield either increasing innovation in step-size as in Grossman-Helpman, or decreasing innovation in step-size.

Second, the competitive innovation model has the extra widening parameter b, representing the rate at which productive capacity increases or is turned into usable output. As knowledge capital becomes easier to reproduce (larger b) the intensity of innovation increases. In a certain sense the Grossman-Helpman model, like all models of this class, assumes that $b = \infty$. This is because once the fixed cost is paid and a new rung has been introduced the technology allows one to make an infinite number of copies of the jth good: a finite number of copies obtains only because the monopoly power of the first innovator is used to prevent competitive imitation. Put it differently, the Grossman-Helpman model assumes that the movement from one particular vintage to the next can be infinitely rapid (once discovered, knowledge is a public good and everyone has it); the presence of a fixed cost at each step explains why things do not go haywire. This is the key difference allowing us to study competitive innovations, something standard models must rule out by hypothesis.

7. Fixed Cost of Knowledge Capital

There can be little doubt as an empirical fact that there is a fixed cost in creating new knowledge. Two left-halves of a blueprint for a new good are scarcely a good substitute for the left and right half. Of course there are fixed costs in producing just about everything, and this has not prevented the competitive model with perfect divisibility from being a useful tool in studying a wide range of market phenomenon. In fact, as we showed above, the equilibrium path of our model is such that it looks like an underlying fixed cost exists. During the build-up phase a large stock of knowledge capital of quality j is accumulated to be turned at once into new knowledge capital $j+1$ when the growth phase starts. Never-the-less, it would be reassuring to know that the explicit introduction of fixed costs into our model of competitive innovation does not cause things to collapse.

Let us assume, then, that there is a fixed cost of introducing new knowledge for the first time. Specifically, to produce for the first time quality $j+1$ knowledge capital from quality j knowledge capital requires a fixed cost of F units of quality j knowledge capital. This results in the creation of $\bar{k} < F$ initial units of quality $j+1$ knowledge capital.[8] For computational simplicity and notational convenience, once the fixed cost is incurred, we assume that it is possible to convert additional units of quality j

[8] If it were possible to convert new knowledge capital into old at the same ratio as in the forward conversion, then the fixed cost would not matter, as it could be "undone" by a subsequent backward conversion. This may partially explain why, when we consider the empirics of fixed costs, they do not seem to matter a great deal: most electrical engineers, if not most economists, can still change a light bulb by themselves.

knowledge capital to quality $j+1$ knowledge capital at the same rate $F/\bar{k} = a$. This may seem a strong assumption – it might seem plausible that, after new knowledge is first introduced, the cost of converting additional old knowledge into the former should fall. We will show later that, properly modeled, such an assumption adds complication to the model, without changing the essential results.

We will assume throughout that

$$\bar{k} \leq k^* \equiv \frac{a^{\frac{b}{b-\rho}} - 1}{a^{\frac{b}{b-\rho}} - a} > 1\,.$$

To understand this assumption, observe that $k^* > 1$ and there is at most one unit of labor available to produce consumption. If $\bar{k} > 1$ then the first installment saturates the entire market for new knowledge capital. In fact, if $\bar{k}$ is even larger, $\bar{k} > k^*$, then we can show that any attempt to innovate must result in the price of the newly created capital falling to zero, meaning that innovation will not take place under competition. This is often taken as the classical case - the case that Romer, Grossman and Helpman, Aghion and Howitt and others apparently have in mind. But it seems implausible to us that the first blueprint will saturate the market for blueprints, that nobody would want even a single additional copy. Of course these subsequent copies of blueprints might cost considerably less than the original. This is the case in our model, since blueprints devoted to copying grow at the rate b, and the price of blueprints fall exponentially over time.

It is convenient to take as parameters F, a. This enables us to compare the indivisible case more directly with the divisible case. Implicitly it means that we vary $\bar{k}$ as F changes. So our assumption on $\bar{k}$ means that

$$F \leq \frac{a^{\frac{b}{b-\rho}} - 1}{a^{\frac{\rho}{b-\rho}} - 1}\,.$$

In the model without fixed cost, the exact time at which quality j knowledge capital is converted into the first unit of quality $j+1$ knowledge capital is a matter of indifference. Any quality j capital not being used to produce consumption may equally well be used to produce more capital of the same quality to be converted to a higher quality at a later date, or can equally well be converted right now. Amongst all these equilibria, there is one along which capital of quality j is not converted to capital of quality $j+1$ until the first moment at which quality $j+1$ is used for the first time in the production of consumption – that is, at the end of build-up and beginning of growth. Moreover, there is a unique such equilibrium in which all quality j capital not being used to produce consumption is converted to quality $j+1$ at that time. Along the equilibrium path, let F^* denote the amount of quality j capital not being used in the production of consumption at the end of build-up, and let t be the time at which that build up ends. Then $F^* = k_{jt} - 1$. Since the equilibrium is a stationary two-period cycle, this is independent of t, and in the Appendix we show that

$$F^* = \frac{a^{\frac{\rho}{b-\rho}}}{a^{\frac{\rho}{b-\rho}} - 1} \frac{\lambda^{\frac{\rho}{b-\rho}} - 1}{\lambda^{\frac{\rho}{b-\rho}}} \frac{a-1}{\lambda - 1} \frac{b-\rho}{\rho}.$$

We are led then to consider two cases: the case of small fixed cost in which $F \le F^*$, and the case of large fixed cost in which $F > F^*$.

Small Fixed Cost

First, consider the case of small fixed cost. In this case there is an equilibrium of the divisible economy in which the fixed cost constraint does not bind. That is, there is an equilibrium in which no one chooses to convert quality j knowledge capital to quality $j+1$ until the end of the build-up, at which time the amount converted is greater than the fixed cost of innovation. Put differently, the fixed cost constraint – that capital must be converted in a minimal amount – is irrelevant. This gives us a minimal robustness property – if the fixed cost is small enough, nothing changes, and competition is free to work its magic.

8. Large Fixed Cost

We now examine the case where the fixed cost of innovation is large in the sense that $F > F^*$. In other words, the divisible equilibrium is no longer feasible, because there is insufficient quality j knowledge capital available to meet the fixed cost of innovation at the time when quality $j+1$ knowledge capital is first to be used to produce consumption. However, even though innovation is not possible at the time at which build-up would usually end, that is at t^b, as time continues to pass and consumption remains constant, capital will continue to grow, so there is a later time at which it will be feasible to pay the fixed cost. For the sake of discussion, suppose the first time this happens is $t > t^b$. Clearly, innovating at t is not necessarily a competitive equilibrium. Why not? In the divisible case, it is possible to introduce a small amount of quality $j+1$ knowledge capital at an intermediate time $t^b < t' < t$ and earn a profit, so innovating "late" is not a competitive equilibrium. With a fixed cost, it is no longer possible to introduce a "small amount" of quality $j+1$ knowledge capital – it is necessary to introduce a large amount F, and innovating "late" becomes a possible competitive equilibrium.

Our analysis of equilibrium with fixed costs, then, will proceed in two stages. First, we will drop from the definition of competitive equilibrium the requirement that an "early" innovation at a time $t^b < t' < t$ not generate profit at existing equilibrium prices. The resultant equilibrium we describe as an *atomistic equilibrium*, meaning that because individual competitors are too small to introduce an innovation on their own they cannot take advantage of a profit opportunity from innovating, even if one exists. Not surprisingly, there are a great many of these equilibria – although, surprisingly, they have a great deal of similarity to each other. After analyzing atomistic equilibria, we will turn attention to *entrepreneurial equilibrium* in which there are entrepreneurs who can raise the funds necessary to innovate by themselves, and who recognize that their decision to innovate will change equilibrium prices.

Atomistic fixed cost equilibrium

An *atomistic fixed cost equilibrium*, or more briefly, an atomistic equilibrium, is a competitive equilibrium minus the requirement that potential innovation bring non-positive profits – the requirement that actual innovation bring zero profits, of course, remains unchanged. Our analysis of the pricing of knowledge capital remains valid. Now, however, we must allow the possibility of elongating the build-up phase so that innovation takes place "late" compared to the competitive equilibrium. Analyzing these equilibria, we show that the zero-profit on innovation condition forces a jump in consumption when innovation takes place. Although there are many equilibria, we show that the combined length of the build-up and growth phases remains the same as in the divisible case. In this sense "plus ça change, plus cest la même chose." There is, however, one remaining class of atomistic equilibria along which innovation may stop altogether. This must be an equilibrium as there are no entrepreneurs able to contemplate introducing a new idea into an equilibrium without innovation, Subsequently, we show that the presence of entrepreneurs eliminates all but one of the atomistic equilibria – and in particular eliminates equilibria in which innovation stops entirely.

The key to characterizing atomistic equilibria is to analyze the zero-profit condition on innovation. Suppose at a particular time t – which we may wish to think of as the end of growth – a single unit of quality j knowledge capital is used to produce consumption, while a moment earlier both qualities j and $j-1$ were used. Then from our analysis of the price of knowledge capital

$$q_{jt} = \frac{\lambda^j - \lambda^{j-1}}{b(1-1/a)}\frac{1}{c_t} = \frac{\lambda^j - \lambda^{j-1}}{b(1-1/a)}\frac{1}{\lambda^j}.$$

Consider the subsequent build-up. The price of quality j capital falls according to

$$q_{j,t+\tau} = \frac{\lambda^j - \lambda^{j-1}}{b(1-1/a)}\frac{1}{\lambda^j}e^{-(b-\rho)\tau}.$$

When quality $j+1$ is used to produce consumption for the first time, its price must be

$$q_{j+1,t+\tau} = \frac{\lambda^{j+1} - \lambda^{j}}{b(1-1/a)}\frac{1}{c_{t+\tau}} = aq_{j,t+\tau}.$$

This we may solve to find the condition for zero profit in innovation.

$$c_{t+\tau} = \frac{\lambda^{j+1}}{a}e^{(b-\rho)\tau}.$$

We must also have $c_{t+\tau} \geq \lambda^j = c_t$, since using a higher quality knowledge capital to displace a lower quality must necessarily increase the amount of consumption produced. In the case without fixed costs, this must hold with exact equality, yielding our old value for $\tau^b = \log(a/\lambda)/(b-\rho)$. If not, that is, if $c_{t+\tau} > \lambda^j$, then at the new value for τ

$$\frac{\lambda^{j+1}}{a}e^{(b-\rho)\tau} > \lambda^j$$
$$\frac{\lambda}{a} > e^{-(b-\rho)\tau}$$

A moment before τ, when consumption was still $c_t = \lambda^j$, the value of quality j capital in producing consumption was

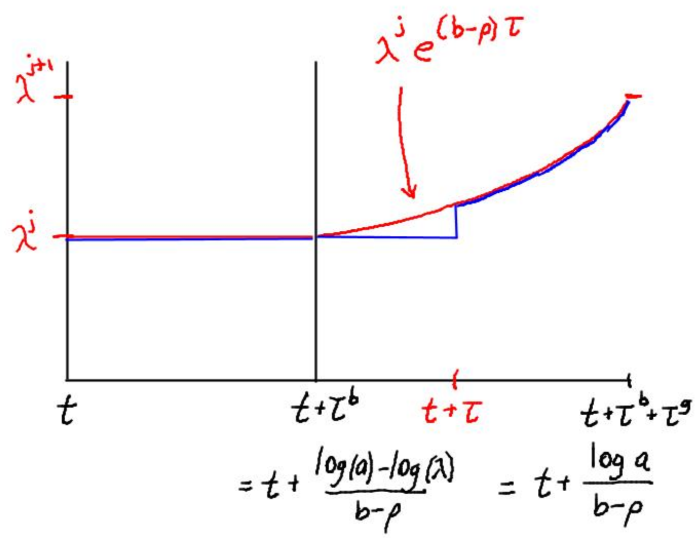
$$q_{j+1,t+\tau-} = \frac{\lambda^{j+1} - \lambda^j}{b(1-1/a)} \frac{1}{\lambda^j} > \frac{\lambda^{j+1} - \lambda^j}{b(1-1/a)} \frac{a}{\lambda^{j+1}} e^{-(b-\rho)\tau} =$$
$$= a \frac{\lambda^j - \lambda^{j-1}}{b(1-1/a)} \frac{1}{\lambda^j} e^{-(b-\rho)\tau} = a q_{j,t+\tau-}$$

meaning that there would be a profit from introducing a small amount of quality $j+1$ knowledge capital.

In an atomistic equilibrium we drop the requirement that there be no (potential) profit from introducing a small amount of innovation. So it is now possible that innovation occurs at $\tau > \tau^b = \log(a/\lambda)/(b-\rho)$, at which time, a discrete jump in consumption from λ^j to

$$c_{t+\tau} = \frac{\lambda^{j+1}}{a} e^{(b-\rho)\tau}$$

takes place. Notice that consumption jumps to the same level it would have been at had innovation taken place at τ^b, and grown at the same rate between then and τ. In other words, regardless of where the jump takes place, all of these paths share the same combined length of build-up and growth, and the same innovation intensity $j^* = (b-\rho)/\log(a)$. A consumption path corresponding to a particular jump point is



illustrated in Figure 1.

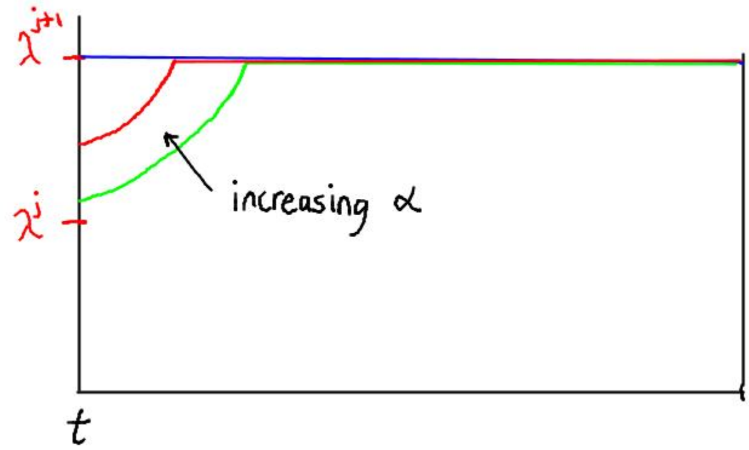
We now have candidates for atomistic equilibria, but we must check whether they are feasible given the fixed cost. Consider a consumption jump of size $1 < \alpha < \lambda$, where $c_{t+\tau} = \alpha c_t = \alpha\lambda^j$. When the jump takes place, we can determine the amount of quality j knowledge capital required to produce consumption from the equation

$$\lambda^j d_{j,t+\tau} + \lambda^{j+1}(1 - d_{j,t+\tau}) = \alpha\lambda^j .$$

Call the solution $d(\alpha) = (\lambda - \alpha)/(\lambda - 1)$. Suppose that at this time all remaining capital of quality j is converted to $j+1$. Call this amount $F*(\alpha)$ and suppose that, as additional quality j is freed as consumption and the stock of $j+1$ increase, it is also converted to quality $j+1$. Once growth ends, and buildup begins, the stock of quality $j+1$ capital continues to grow. If $F*(\alpha)$ was chosen appropriately, at the beginning of the next jump the amount of quality $j+1$ knowledge capital will be exactly $d(\alpha) + F*(\alpha)$ meaning that we are in a steady state cycle. In the Appendix, we compute

$$F*(\alpha) = \frac{-1 - \frac{1}{\lambda - 1}\left(a^{\frac{b}{b-\rho}} - \lambda a^{\frac{\rho}{b-\rho}}\right) - \frac{a-1}{\rho}\frac{a^{\frac{\rho}{b-\rho}}}{\lambda - 1}\left((b-\rho)\lambda\left(\frac{\alpha}{\lambda}\right)^{\frac{b}{b-\rho}} - b\alpha\right)}{a^{\frac{\rho}{b-\rho}} - 1} - \frac{\lambda - \alpha}{\lambda - 1} .$$

There it is also proven that this is increasing in α, which can easily be seen by examining the path of consumption from the beginning of growth to the end of the buildup phase, as shown below in Figure 2



As α increases the total amount of consumption over the cycle is increased, meaning that, for any initial amount of knowledge capital, less will be left at the end. So the higher the value of α, the more quality j capital is needed at the beginning of the cycle to have a steady state cycle. Moreover, the larger is α, the smaller is $d(\alpha)$, meaning that less must be left over after conversion, so we see that $F*(\alpha)$ must necessarily increase.

From the perspective of fixed cost, what is essential is that $F \leq F*(\alpha)$. That is, the amount of capital that can be converted in any case is at least as great as the amount that must be converted due to fixed cost. Since $F*(\alpha)$ is increasing in α, larger jumps mean that it is possible to sustain an equilibrium with a larger fixed cost. Morever, we show in the Appendix that

$$F*(\lambda) = \frac{a^{\frac{b}{b-\rho}} - 1}{a^{\frac{\rho}{b-\rho}} - 1},$$

so that under our assumption on $\bar{k}$, there is always some value of α such that $F \leq F(\alpha)$. In other words, steady state atomistic equilibria with continued innovation and innovation intensity $j* = (b-\rho)/\log(a)$ exist.

Entrepreneurial Equilibrium

We turn now to *entrepreneurial equilibrium* in which there are competing entrepreneurs who have the ability to make discrete changes to the stock of knowledge capital and who understand how this will cause prices to change. We abstract from any (even instantaneous) monopoly power that entrepreneurs may have acquired by being first to innovate. We assume the entire amount of new knowledge capital created – $\bar{k}$, at a minimum – must be sold, and that, once this is done, it trades in a perfectly competitive market.

Specifically, we wish to consider the condition that there should be no profit from deviation by innovating "early," that is before the invention was to take place on the equilibrium path. Since there are many potential innovators, one along each moment, it is not possible for any individual to force a delay in the time of innovation. Our goal is to establish that – unlike atomistic equilibrium – there is a unique entrepreneurial equilibrium. We will show, moreover, that the unique entrepreneurial equilibrium is the steady state atomistic equilibrium in which innovation occurs as early as possible, that is, $F^*(\alpha) = F$.

For simplicity, we will consider only deviations that take place when a single quality of capital is being used to produce consumption and there is full-employment, that is, during buildup. A weak justification for this assumption is that it requires that deviations not be "too large" in the sense that the assumption does not allow innovation "too" much earlier than it was to take place originally. It also rules out a second deviation for a period of time after the first deviation. More to the point, intuition suggests that it is unlikely that it would be profitable to introduce a new quality of knowledge capital while it is still possible to increase consumption by displacing low quality knowledge capital.

In any case, a formal analysis of the case in which early innovation is possible even before buildup begins is enormously complicated. With the assumption that deviations can take place only when a single quality of knowledge capital is in use, we need consider only atomistic equilibria which start with two qualities of knowledge capital. Without this assumption, there is no limit on the number of qualities that can be introduced in a short period of time, and we would need to analyze atomistic equilibria with arbitrary initial combinations of qualities of knowledge capital. In our view, this vast increase in the complexity of the analysis adds little to our economic understanding, so the primary reason for making this assumption is simplicity. We will highlight below, the one place we use this assumption.

A deviation consists of buying quality j knowledge capital and converting at least F of it to quality $j+1$ at some time t. Because prices are arbitraged, the specific time at which the quality j knowledge capital is purchased does not matter, so we can assume it is purchased at the same time conversion takes place. We also allow the conversion of more knowledge capital than exists: by waiting a sufficiently long time, it is possible to accumulate large amounts of extra knowledge capital by removing small amounts from the production of consumption at each moment of time. We assume, in other words, the entrepreneurs are price takers in input markets.

That said, there are in fact two different input market prices to be considered. That is, it may seem natural to assume that an entrepreneur can acquire knowledge capital at the existing price in the old equilibrium, especially if this is accomplished over an interval of time by purchasing small quantities in each istant. However, the decision to innovate will not only cause the price of the new quality $j+1$ knowledge capital to change, it will generally change also the price of quality j knowledge capital. Owners of that knowledge capital may observe the entrepreneur acquiring knowledge capital, anticipate his decision to innovate, and this will change the price that they charge. Suppose q_{jt} is the existing equilibrium price of knowledge capital, and q_{jt}^D is the price in the new equilibrium after the deviation. If $q_{jt} < q_{jt}^D$ existing holders of capital of type j infer from the request to purchase that they should anticipate this capital gain, so they should demand q_{jt}^D for their existing capital. On the other hand, the desire to sell capital does not indicate a shift in equilibrium, so everyone stands willing to buy capital at q_{jt}. If there is to be a capital loss, then, the input must be acquired at the old price. If there is to be a capital gain, we assume that the innovator can garner some tiny fraction of the capital gain. So the condition for a *profitable deviation* is that

$$aq_{jt}^D \le q_{j+1,t}^D$$

$$aq_{jt} < q_{j+t,t}^D$$

that is, there cannot be a loss at the new prices, and there must be a strict gain at the old prices. The sale of quality $j+1$, of course, always takes place at the new prices.

This issue of capital gains and losses is not a new one for the theory of innovation: it was first explored in Hirshleifer [1971] who pointed out the possibility of very great profits from prior trading based on inside information about a soon to occur innovation. Here knowledge is common, so such a "secret" deviation is not possible – in a model where innovation was less competitive it would be a crucial issue, and a "competitive" theory could well result in too much innovation rather than too little, as Hirshleifer pointed out.[9]

We next need to consider what price should be expected to occur after a deviation. Certainly we should expect at a minimum that following a deviation an atomistic equilibrium would occur. But which one? One answer is to follow the logic of subgame perfection, and simply say that an atomistic equilibrium is *weakly entrepreneurial* if for any deviation there is some atomistic equilibrium starting from that initial condition that is unprofitable. To be formal, let E be a set of atomistic equilibria We say that E is *comprehensive* if it has a continuation equilibrium for any initial condition that has no more than two adjacent qualities of knowledge capital and that any continuation of a path in E lie in E. Any subset E of the set of atomistic equilibria that is comprehensive and such that for each $e \in E$ and each deviation there exists a $e' \in E$ which makes the deviation unprofitable we call *weakly entrepreneurial.*

Although this notion is too weak to enable us to pin down a particular equilibrium, it is enough to rule out equilibria in which innovation never occurs. That is, all weakly entrepreneurial equilibria share the same innovation intensity.

In the theory of entrepreneurial equilibria the steady state atomistic equilibrium in which innovation occurs as early as possible, that is, where $F^*(\alpha) = F$, plays a special role. Call this the *earliest atomistic equilibrium*, and denote it by e^*. Our first result is a crucial technical analysis of the stock of knowledge capital in atomistic equilibria. Let $k^b(\alpha)$ denote the knowledge capital stock needed at the beginning of build-up required to sustain a steady state atomistic equilibrium with a consumption jump of size α.

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[9] A similar issue arises in ordinary competitive equilibrium theory, although it is not widely recognized. In general why not "buy a monopoly" or "corner the market" by purchasing the entire existing stock of capital and destroying part of it? Here premature innovation has some of the flavor of "destroying capital." Buying at the competitive price, and selling at the monopoly price is profitable. One answer is the one we give here: when a trader deviates and, at the going equilibrium price, tries to corner the market, other traders demand a higher than competitive price in anticipation of subsequent "monopolization."

Lemma 1: *If* $\alpha(F)$ *corresponds to the earliest atomistic steady state and* t^b *is the beginning of build-up in an atomistic equilibrium where there is only quality* j *knowledge capital, then* $k_{jt^b} \leq k^b(\alpha(F))$.

Proof: Fix an atomistic equilibrium, and let $t_1, t_2, \ldots, t_j, \ldots$ be the times at which build-up phases corresponding to quality j knowledge capital begin.

First: $k^b(\alpha)$ is strictly decreasing

Second: let α be the inf of α_j, where α_j is the jump in consumption between the quality j build-up and growth phase

Third: for some t_j $k_{jt^b} > k^b(\underline{\alpha})$ (choose α_j very close to α. Then $k_{jt^b} > k^b(\alpha_j)$ by some decent margin, since we have to convert a full $F > F^*(\alpha_j)$ while still leaving behind $(\lambda - \alpha_j)/(\lambda - 1)$ units of quality j.

Fourth:

$$k_{jt_j} - k^d(\underline{\alpha}) \geq a^{\frac{1}{b-r}}(k_{j-1,t_{j-1}} - k^d(\underline{\alpha}))$$

So capital goes to infinity violating the transverality condition.

☑

We first establish that the earliest equilibrium, at least, is weakly entrepreneurial.

Lemma 2*: There is no profitable deviation from the earliest atomistic equilibrium to any atomistic fixed price equilibrium.*

Next we show that we can, at least, rule out the atomistic equilibria in which innovation stops.

Lemma 3*: If* E *is comprehensive and* $e \in E$ *stops innovating after a time* t *then there is a deviation that is profitable for any continuation* $e' \in E$ *..*

Proof: Jump directly to the maximum capital stock $k^b(\alpha(F))$. The only equilibrium that starts there is e^*, so since E is comprehensive, $e^* \in E$. If we waited long enough that the price of existing capital is trivially small, then clearly e^* is profitable. There is zero profit at e^* prices since no unemployment, and profit at e prices since price of existing capital is trivially small.

☑

Here point out why the non-innovating equilibria that are not entrepreneurial when $\bar{k} \leq k^*$ are in fact entrepreneurial, and indeed the only entrepreneurial equilibria when $\bar{k} > k^*$. The continuation equilibrium following any deviation necessarily has zero price of all kinds of capital.

More generally there can be multiple self-reinforcing "punishment" equilibria traders effectively "retaliate" against innovation by lowering their demand for knowledge capital. However, in our view, this is not a very interesting assumption. The entrepreneur not only creates new knowledge capital, but also can at least set the initial price at which he is willing to sell that capital. Hence, in effect, he can choose which equilibrium follows the deviation. The complication is that in making this choice, he should not be free to choose any atomistic equilibrium, but only one which is itself robust to further deviations.

To be formal, let E be a set of atomistic equilibria We say that $e \in E$ is *blocked* by $e' \in E$ if there is a profitable deviation from e that results in the continuation equilibrium e'. Any subset E of the set of atomistic equilibria that is comprehensive and for which no equilibrium is blocked by another is called a set of *strongly entrepreneurial equilibria*.

The notion that innovators are "optimistic" in their beliefs in the sense that they believe that they get the most favorable equilibrium might seem contrary to the spirit of subgame perfection. However, applying the concept to Markov games, if is in fact simply an extension of the idea of non-retaliation implicit in Markov perfect equilibrium. In particular, the set of continuations that arises in a Markov perfect equilibrium is entrepreneurial in our sense: since after each initial condition there is only one possible continuation, "optimism" is irrelevant – there is nothing for the entrepreneur to pick between. In other words, we require that a profitable deviation itself be entrepreneurial – and this means that existence is not a major issue, since there are general conditions for the existence of Markov perfect equilibria. Hence in each period, each player must be optimizing. Again, this emphasizes that the notion of entrepreneurial equilibrium is the right generalization of the idea that players continue to do what they would have done anyway in the face of a deviation – they do not attempt to punish innovators.

We will show that the earliest atomistic equilibrium is the unique strongly entrepreneurial equilibrium.

Lemma 4: *if $e \neq e^*$ and e continues innovating, then e is blocked by a deviation that returns to e at the end of the growth phase following an innovation.*

9. Robustness

The basic model is quite special – many assumptions are made for ease of exposition and solution. It is important to understand whether the basic conclusion that robust innovation takes place under competition is sensitive to these assumption.

Labor Cost of Producing Knowledge

We have assumed *contra* Grossman-Helpman that labor is not an input into the knowledge creation process. On the one hand, we doubt that the type of labor used in knowledge creation is a particularly good substitute for the labor used to produce output, so we do not view the Grossman-Helpman assumption as especially realistic either. But what happens if we require some sort of labor or other input into the knowledge creation process?

- Divisible case – obviously doesn't matter
- Fixed cost case – want to produce new knowledge gradually. But the right place to put the fixed cost is in $\bar{k}$ not F. That is, new knowledge can be accumulated gradually, but is not useful until a threshold is crossed. So we need only that knowledge capital is not employed in producing output until after we cross $\bar{k}$ – basically nothing changes.

Spillover Externalities

That is, you just get a fraction of the knowledge capital you produce – the rest gets "spilled over" to other people. Point out that this delays but does not kill innovation, provided the fraction you get is bigger than zero. Point out that it also doesn't make sense that the spillover should be especially large.

Cost of Converting Old Knowledge

If the cost just drops after we get the initial unit, then we are dead – this is like a 100% spillover externality. But it makes not sense: we can distinguish two techniques: learning from scratch, and being taught. The former doesn't get easier just because someone else already learned from scratch. The latter requires the new knowledge as input. So add an activity – using new knowledge to convert old knowledge. Show what happens in a simple model with a target quantity of new knowledge in the divisible case. First you surge up the knowledge using the learning from scratch activity, then you start converting. But you don't create some infinitesimal amount of new knowledge from scratch then switch over – this would give you very little capacity for conversion, so wouldn't get you to the target.

10. Conclusion

In our account of innovation, fixed costs and monopoly power play at best a peripheral role. In the traditional account they are at center stage. This is because, we imagine, a misunderstanding that accounting for endogenous innovation and growth requires increasing returns to scale; that the source of increasing returns to scale is fixed cost; and that without monopoly fixed costs are an insurmountable obstacle to production. For this reason, little evidence about the importance of fixed costs is ever examined. Yet fixed costs and the indivisibilities that give rise to them are ubiquitous. Virtually every type of manufacturing industry has a minimum plant size; human beings come in single indivisible units, and even such humble commodities as shoes can not be produced as less than a single pair, nor wheat less than a single grain. Yet we acknowledge that most of these indivisibilities are not terribly important.

In the traditional Grossman/Helpman setup, we may imagine that one-period monopoly is a convenient analytic device for reflecting the fact that the producers cannot appropriate the entire surplus of his indivisible invention. Yet we may ask in practice whether private institutions might not arise that enable enough recovery of cost that the social optimum might not be a better description of reality than one-period monopoly. The assumptions that Grossman and Helpman use to solve the model are at best questionable. The model is driven by patents that last until the next invention then disappear – not something very close to reality. The first mover advantage; competitive rent; downstream licensing of inventions – all these things are assumed away. Moreover, not all industries are covered by patents, and in surveys R&D managers in most industries R&D managers say that patents are unimportant. [cite the surveys here] Moreover, the entire literature indicating that first mover advantages lead to patent races over intensity, and greatly increase innovation are ignored in their model. Finally, absent negotiating costs, private information, other transactions costs, we surely expect that the industry can work out efficient arrangements. In the pharmaceutical industry where there are a few large purchasers – is it unimaginable that they could work out a satisfactory long-term contract? The Grossman/Helpman assumptions are sensible for giving a simple way to solve the model – but should not be taken seriously as a comment on the efficiency of equilibrium in practice. Of course, if the simple one-period patent mechanism was relatively efficient we might see it in practice – [give the Grossman-Helpman figure where there is a broad range of lambda where things are pretty good.] But if was dreadfully inefficient, would we expect it to be so persistent? Or would new institutions arise?

If fixed cost are potentially unimportant even in the traditional account of innovation, they are so much less so in ours account of innovation, small fixed cost is irrelevant completely – and even large fixed cost has no impact on the intensity of innovation.

It is sensible to ask, then, how important are fixed costs in practice. To the extent that they are small, we may reasonably suppose that they are no more central to innovation, than, say to the study of the production of shoes. As one example, we might cite the automobile industry. From Business Week we learn the basic facts about this industry. First, the cost of a new model is quite large: $400 million – on the same order as the cost of a major new drug – including the cost of failures, and much more than, for example, the cost of a high budget movie. Wikepedia reports that there are 14 major automobile producers world-wide – and in the major markets, the United States especially, they compete on a relatively level footing. Finally, patent and intellectual property protection for automobile designs is essentially non-existent. Models, once introduced may be copied with a lag of about a year, yet once in production, major changes occur roughly only every xxx years. So based on the world of Grossman and Helpman – we might expect not to see new models introduced – yet we do.

Turning next to the example of DRAM with which we started. From Park [2002] we learn: “Typically, each generation of DRAM is introduced by a leading firm, induces (sequential) entries of up to 20 producers...Usually, the firms, which succeed in the innovation...come up with various different physical designs...” In other words, intellectual property does not play much of a role in this industry and there are a lot of producers. There is also a substantial fixed cost of building new “fabs” in which to produce chips. What happens in this market is that the first-movers get to sell early in the market where output is low and price high. This is in fact a competitive rent, and not due to monopoly at all: output is low and price high because capacity is low and marginal utility is high, as in our model. In particular, a producer would be foolish to artificially hold down production in the early periods where demand is elastic, and everyone is racing to beat them out. [Reference our rent seeking paper.]

Yet more evidence for the relative unimportance of short-term monopoly can be found in the common practice of patent pools [reference Shapiro here]. For example, the steel patent pool [exact details of pool here] was formed not to inhibit innovation, but because innovation was at a standstill. On top of this, there is almost total lack of evidence that patents have any effect on the rate of innovation. [reference Lerner here]

We should mention that one of the purposes of the Grossman-Helpman paper was to show how a model of a quality ladder is equivalent to one in which new goods are introduced, even as old ones continue to be used. The model of product diversity is a perfectly sensible account of the benefit of innovation – but it does not make any important difference to the main story. If there are incentives to innovate when a technology has played out, whether it will continue to be used after innovation is not crucial. In some cases the model of technological replacement is the relevant one. For example computer hardware since the Macintosh has had the same basic elements of screen, mouse, removable storage, cpu, and a disk drive. Yet each has undergone numerous upgrades. The flat screen replaced the CRT, the recordable cd replaced the floppy. On the other hand, the laptop has supplemented but not replaced the desktop. In software the so called “killer apps” have tended to supplement but do not replace each other: the spreadsheet, was not replaced by the word processor, which was not replaced by email, which was not replaced by the web browser, which was not replaced by p2p networking, which was not replaced by instant messaging, which in turn was not replaced by Google. On the other hand, within categories innovation proceeds via upgrades. In spreadsheets: Visicalc was replaced by Lotus 123, and then Excel. In word processing Wordstar was replaced by Wordperfect, then Word. But the bottom line is the question of when people start looking for and talking about the "next killer app." In the software industry, as in our model, this happens exactly when the last killer app is not so killer any more.

Appendix: Steady State Capital

Consider a growth phase beginning at time 0. Suppose that initial knowledge capital is entirely of quality j, and that there is initially k_{j0} units of that quality. Suppose that consumption jumps immediately to c_0. It will be convenient to define $\xi_t = c_t / \lambda^j$. At time t consumption has grown to $c_0 e^{(b-\rho)t}$ until it reaches λ^{j+1} ending the growth phase at

$$t^g = \tau^g = \frac{\log(\lambda) - \log(\tilde{c}_0)}{b - \rho}.$$

During this time we assume that quality j knowledge capital is converted to quality $j+1$ as soon as it is freed from use in producing consumption. Specifically,

$$\begin{aligned} k_{jt} = d_{jt} &= \frac{\lambda^{j+1} - c_t}{\lambda^{j+1} - \lambda^j} = \frac{\lambda - \xi_t}{\lambda - 1} \\ &= \frac{\lambda - e^{(b-\rho)t}\xi_0}{\lambda - 1} \end{aligned}.$$

So

$$\begin{aligned} h_{jt} &= -\dot{k}_{jt} = -\dot{d}_{jt} \\ &= \frac{(\dot{c}_t / \lambda^j)}{\lambda - 1} = (b - \rho)\frac{(c_t / \lambda^j)}{\lambda - 1} \\ &= (b - \rho)e^{(b-\rho)t}\,\frac{\xi_0}{\lambda - 1} \end{aligned}$$

We find then that quality $j+1$ knowledge capital grows according to

$$\begin{aligned} \dot{k}_{j+1,t} &= b(k_{j+1,t} + d_{jt} - 1) + \frac{b - \rho}{a}e^{(b-\rho)t}\,\frac{\xi_0}{\lambda - 1} \\ &= b(k_{j+1,t} + \frac{1 - e^{(b-\rho)t}\xi_0}{\lambda - 1}) + \frac{b - \rho}{a}e^{(b-\rho)t}\,\frac{\xi_0}{\lambda - 1} \\ &= b(k_{j+1,t} + \frac{1}{\lambda - 1}) + \frac{(1-a)b - \rho}{a}e^{(b-\rho)t}\,\frac{\xi_0}{\lambda - 1} \end{aligned}$$

until the end of the phase. At the end of build-up at t^b, quality $j+1$ knowledge capital is $k_{j+1,t^b} = 1 + e^{b\tau^b}\left(k_{j+1,t^g} - 1\right)$. Since $\tau^g + \tau^b = \log a / (b - \rho)$, we have

$$\tau^b = \frac{\log(a) - \log(\lambda) + \log(c_0 / \lambda^j)}{b - \rho},$$

or

$$k_{j+1,t^b} = 1 + \left(\frac{a\xi_0}{\lambda}\right)^{\frac{b}{b-\rho}} \left(k_{j+1,t^g} - 1\right).$$

We next guess a solution to

$$\dot{k}_{j+1,t} = b(k_{j+1,t} + \frac{1}{\lambda - 1}) + \frac{(1-a)b-\rho}{a} e^{(b-\rho)t} \frac{\xi_0}{\lambda - 1}$$

of the form $k_{j+1,t} = a_0 + a_1 e^{bt} + a_2 e^{(b-\rho)t}$. Then by differentiating

$$\dot{k}_{j+1,t} = ba_1 e^{bt} + (b-\rho)a_2 e^{(b-\rho)t} ,$$

while substituting into the differential equation gives

$$\begin{aligned} \dot{k}_{j+1,t} &= b(a_0 + a_1 e^{bt} + a_2 e^{(b-\rho)t} + \frac{1}{\lambda - 1}) + \frac{(1-a)b-\rho}{a} e^{(b-\rho)t} \frac{\xi_0}{\lambda - 1} \\ &= \left(ba_0 + \frac{b}{\lambda - 1}\right) + ba_1 e^{bt} + \left(ba_2 + \frac{(1-a)b-\rho}{a} \frac{\xi_0}{\lambda - 1}\right) e^{(b-\rho)t} \end{aligned}.$$

We conclude that

$$ba_0 + \frac{b}{\lambda - 1} = 0$$

and

$$(b-\rho)a_2 = ba_2 + \frac{(1-a)b-\rho}{a} \frac{\xi_0}{\lambda - 1}.$$

This gives

$$a_0 = -1/(\lambda - 1)$$

and

$$a_2 = -\frac{(1-a)b-\rho}{a\rho} \frac{\xi_0}{\lambda - 1}.$$

Plugging back in to our guessed solution

$$k_{j+1,t} = -\frac{1}{\lambda - 1} + a_1 e^{bt} - \frac{(1-a)b-\rho}{a\rho} \frac{\xi_0}{\lambda - 1} e^{(b-\rho)t}.$$

The initial condition is

$$k_{j+1,0} = \frac{k_{j0} - d_{j0}}{a} = \frac{k_{j0} - \dfrac{\lambda - \xi_0}{\lambda - 1}}{a},$$

so we find

$$\frac{k_{j0} - \dfrac{\lambda - \xi_0}{\lambda - 1}}{a} = -\frac{1}{\lambda - 1} + a_1 - \frac{(1-a)b - \rho}{a\rho}\frac{\xi_0}{\lambda - 1},$$

enabling us to find the missing coefficient

$$a_1 = \frac{k_{j0} - \dfrac{\lambda - \xi_0}{\lambda - 1}}{a} + \frac{1}{\lambda - 1} + \frac{(1-a)b - \rho}{\rho a}\frac{\xi_0}{\lambda - 1}.$$

Consequently

$$k_{j+1,t} =$$

$$-\frac{1}{\lambda - 1} + \left(\frac{k_{j0} - \dfrac{\lambda - \xi_0}{\lambda - 1}}{a} + \frac{1}{\lambda - 1} + \frac{(1-a)b - \rho}{\rho a}\frac{\xi_0}{\lambda - 1} \right) e^{bt} - \frac{(1-a)b - \rho}{a\rho}\frac{\xi_0}{\lambda - 1} e^{(b-\rho)t}$$

The end of the growth phase occurs at

$$t^g = \tau^g = \frac{\log(\lambda) - \log(\xi_0)}{b - \rho}$$

so $e^{(b-\rho)t^g} = \lambda / \tilde{c}_0$ and

$$e^{bt^g} = \left(\frac{\lambda}{\xi_0} \right)^{\frac{b}{b-\rho}}.$$

Plugging in we find

$$\begin{aligned} k_{j+1,t^g} &= -\frac{1}{\lambda - 1} + \left(\frac{k_{j0} - \dfrac{\lambda - \xi_0}{\lambda - 1}}{a} + \frac{1}{\lambda - 1} + \frac{(1-a)b - \rho}{\rho a}\frac{\xi_0}{\lambda - 1} \right) \left(\frac{\lambda}{\xi_0} \right)^{\frac{b}{b-\rho}} \\ &\quad - \frac{(1-a)b - \rho}{\rho a}\frac{\xi_0}{\lambda - 1}\frac{\lambda}{\xi_0} \\ &= -\frac{1}{\lambda - 1} + \left(\frac{k_{j0} - \dfrac{\lambda - \xi_0}{\lambda - 1}}{a} + \frac{1}{\lambda - 1} + \frac{(1-a)b - \rho}{\rho a}\frac{\xi_0}{\lambda - 1} \right) \left(\frac{\lambda}{\xi_0} \right)^{\frac{b}{b-\rho}} \\ &\quad - \frac{(1-a)b - \rho}{\rho a}\frac{\lambda}{\lambda - 1} \end{aligned}$$

while from above, the terminal capital stock is

$$
\begin{aligned}
k_{j+1,t^b} &= 1+\left(\frac{a\xi_0}{\lambda}\right)^{\frac{b}{b-\rho}}(k_{j+1,t^g}-1) \\
&= 1+\left(\frac{a\xi_0}{\lambda}\right)^{\frac{b}{b-\rho}}\times \\
&(-\frac{1}{\lambda-1}+\left(\frac{k_{j0}-\frac{\lambda-\xi_0}{\lambda-1}}{a}+\frac{1}{\lambda-1}+\frac{(1-a)b-\rho}{\rho a}\frac{\xi_0}{\lambda-1}\right)\left(\frac{\lambda}{\xi_0}\right)^{\frac{b}{b-\rho}} \\
&-\frac{(1-a)b-\rho}{\rho a}\frac{\lambda}{\lambda-1}-1) \\
&= 1+\left(\frac{a\xi_0}{\lambda}\right)^{\frac{b}{b-\rho}}\times \\
&(-\frac{\lambda}{\lambda-1}+\left(\frac{k_{j0}-\frac{\lambda-\xi_0}{\lambda-1}}{a}+\frac{1}{\lambda-1}+\frac{(1-a)b-\rho}{\rho a}\frac{\xi_0}{\lambda-1}\right)\left(\frac{\lambda}{\xi_0}\right)^{\frac{b}{b-\rho}} \\
&-\frac{(1-a)b-\rho}{\rho a}\frac{\lambda}{\lambda-1}) \qquad ,
\end{aligned}
$$

$$
\begin{aligned}
k_{j+1,t^b} &= 1+ \\
&-\frac{\lambda}{\lambda-1}\left(\frac{a\xi_0}{\lambda}\right)^{\frac{b}{b-\rho}}+\left(\frac{k_{j0}-\frac{\lambda-\xi_0}{\lambda-1}}{a}+\frac{1}{\lambda-1}+\frac{(1-a)b-\rho}{\rho a}\frac{\xi_0}{\lambda-1}\right)a^{\frac{b}{b-\rho}} \\
&-\frac{(1-a)b-\rho}{\rho a}\frac{\lambda}{\lambda-1}\left(\frac{a\xi_0}{\lambda}\right)^{\frac{b}{b-\rho}} \\
&= 1+\frac{1}{\lambda-1}a^{\frac{b}{b-\rho}}-\frac{\lambda}{\lambda-1}a^{\frac{\rho}{b-\rho}} \\
&-\frac{\lambda}{\lambda-1}\left(\frac{a\xi_0}{\lambda}\right)^{\frac{b}{b-\rho}}+\left(\frac{\xi_0}{\lambda-1}+\frac{(1-a)b-\rho}{\rho}\frac{\xi_0}{\lambda-1}\right)a^{\frac{\rho}{b-\rho}} \\
&-\frac{(1-a)b-\rho}{\rho a}\frac{\lambda}{\lambda-1}\left(\frac{a\xi_0}{\lambda}\right)^{\frac{b}{b-\rho}} \\
&+a^{\frac{\rho}{b-\rho}}k_{j0}
\end{aligned}
$$

$$
\begin{aligned}
k_{j+1,t^b} &= 1 + \frac{1}{\lambda - 1}\left(a^{\frac{b}{b-\rho}} - \lambda a^{\frac{\rho}{b-\rho}} \right) \\
&\quad - \left(1 + \frac{(1-a)b - \rho}{\rho a}\right) \frac{\lambda}{\lambda - 1} \left(\frac{a\xi_0}{\lambda}\right)^{\frac{b}{b-\rho}} \\
&\quad + \left(1 + \frac{(1-a)b - \rho}{\rho}\right) \frac{a^{\frac{\rho}{b-\rho}}}{\lambda - 1} \xi_0 \\
&\quad + a^{\frac{\rho}{b-\rho}} k_{j0} \\
&= 1 + \frac{1}{\lambda - 1}\left(a^{\frac{b}{b-\rho}} - \lambda a^{\frac{\rho}{b-\rho}} \right) \\
&\quad - \frac{(1-a)(b-\rho)}{\rho} \frac{\lambda a^{\frac{\rho}{b-\rho}}}{\lambda - 1} \left(\frac{\xi_0}{\lambda}\right)^{\frac{b}{b-\rho}} + \frac{(1-a)b}{\rho} \frac{a^{\frac{\rho}{b-\rho}}}{\lambda - 1} \xi_0 \\
&\quad + a^{\frac{\rho}{b-\rho}} k_{j0}
\end{aligned}
$$

$$
\begin{aligned}
k_{j+1,t^b} &= 1 + \frac{1}{\lambda - 1}\left(a^{\frac{b}{b-\rho}} - \lambda a^{\frac{\rho}{b-\rho}} \right) \\
&\quad + \frac{a-1}{\rho} \frac{a^{\frac{\rho}{b-\rho}}}{\lambda - 1} \left((b-\rho)\lambda \left(\frac{\xi_0}{\lambda}\right)^{\frac{b}{b-\rho}} - b\xi_0 \right) \\
&\quad + a^{\frac{\rho}{b-\rho}} k_{j0}
\end{aligned}
$$

So the steady state condition is $k_{j+1,t^b} = k_{j0}$, or

$$
\begin{aligned}
\left(a^{\frac{\rho}{b-\rho}} - 1 \right) k_{j0} &= -1 - \frac{1}{\lambda - 1}\left(a^{\frac{b}{b-\rho}} - \lambda a^{\frac{\rho}{b-\rho}} \right) \\
&\quad - \frac{a-1}{\rho} \frac{a^{\frac{\rho}{b-\rho}}}{\lambda - 1} \left((b-\rho)\lambda \left(\frac{\xi_0}{\lambda}\right)^{\frac{b}{b-\rho}} - b\xi_0 \right)
\end{aligned}
$$

Note that this function is increasing in $\tilde{c}_0$, specifically,

$$\left(a^{\frac{\rho}{b-\rho}}-1\right)\frac{dk_{j0}}{d\tilde{c}_0} = -\frac{a-1}{\rho}\frac{a^{\frac{\rho}{b-\rho}}}{\lambda-1}\left(b\lambda^{-\frac{\rho}{b-\rho}}\xi_0^{\frac{\rho}{b-\rho}}-b\right)$$

$$= -\frac{a-1}{\rho}\frac{a^{\frac{\rho}{b-\rho}}}{\lambda-1}b\left(\left(\frac{\xi_0}{\lambda}\right)^{\frac{\rho}{b-\rho}}-1\right)$$

It follows that

$$F^* = k_{j0} - d_{j0} = k_{j0} - \frac{\lambda-\xi_0}{\lambda-1} = k_{j0} - \frac{\lambda}{\lambda-1} + \frac{\xi_0}{\lambda-1}$$

is also increasing in ξ_0. For reference we can write out the function

$$F^* =$$

$$\frac{-1-\frac{1}{\lambda-1}\left(a^{\frac{b}{b-\rho}}-\lambda a^{\frac{\rho}{b-\rho}}\right)-\frac{a-1}{\rho}\frac{a^{\frac{\rho}{b-\rho}}}{\lambda-1}\left((b-\rho)\lambda\left(\frac{\xi_0}{\lambda}\right)^{\frac{b}{b-\rho}}-b\tilde{c}_0\right)}{a^{\frac{\rho}{b-\rho}}-1} - \frac{\lambda-\xi_0}{\lambda-1}$$

The sanity check is when $\xi_0 = \lambda$, meaning that there is no growth phase

$$\left(a^{\frac{\rho}{b-\rho}}-1\right)k_{j0} = -1-\frac{1}{\lambda-1}\left(a^{\frac{b}{b-\rho}}-\lambda a^{\frac{\rho}{b-\rho}}\right)$$

$$-\frac{a-1}{\rho}\frac{a^{\frac{\rho}{b-\rho}}}{\lambda-1}((b-\rho)\lambda-b\lambda)$$

$$= -1-\frac{1}{\lambda-1}\left(a^{\frac{b}{b-\rho}}-\lambda a^{\frac{\rho}{b-\rho}}\right)$$

$$+(a-1)\frac{\lambda a^{\frac{\rho}{b-\rho}}}{\lambda-1}$$

$$= -1-\frac{1}{\lambda-1}\left(a^{\frac{b}{b-\rho}}-\lambda a^{\frac{\rho}{b-\rho}}\right)$$

$$+\frac{\lambda a^{\frac{b}{b-\rho}}}{\lambda-1}-\frac{\lambda a^{\frac{\rho}{b-\rho}}}{\lambda-1}$$

$$\left(a^{\frac{\rho}{b-\rho}}-1\right)k_{j0}=-1-\frac{1}{\lambda-1}\left(a^{\frac{b}{b-\rho}}-\lambda a^{\frac{\rho}{b-\rho}}\right)$$
$$+\frac{\lambda a^{\frac{b}{b-\rho}}}{\lambda-1}-\frac{\lambda a^{\frac{\rho}{b-\rho}}}{\lambda-1}$$
$$=-1-\frac{a^{\frac{b}{b-\rho}}}{\lambda-1}+\frac{\lambda a^{\frac{b}{b-\rho}}}{\lambda-1}$$
$$=a^{\frac{b}{b-\rho}}-1$$

while the direct calculation is

$$1+\left((k_{j0}/a)-1\right)e^{bt^b}=k_{j0}$$
$$1+\left((k_{j0}/a)-1\right)a^{\frac{b}{b-\rho}}=k_{j0}$$
$$k_{j0}\left(a^{\frac{\rho}{b-\rho}}-1\right)=a^{\frac{b}{b-\rho}}-1$$

Finally, when $\xi_0=1$, that is, when there is no jump in consumption at the start of growth

$$F^*=$$
$$\frac{-1-\frac{1}{\lambda-1}\left(a^{\frac{b}{b-\rho}}-\lambda a^{\frac{\rho}{b-\rho}}\right)-\frac{a-1}{\rho}\frac{a^{\frac{\rho}{b-\rho}}}{\lambda-1}\left((b-\rho)\lambda^{-\frac{\rho}{b-\rho}}-b\right)}{a^{\frac{\rho}{b-\rho}}-1}-1=$$
$$\frac{a^{\frac{\rho}{b-\rho}}\frac{1}{\lambda-1}\left(-(a-\lambda)-\frac{a-1}{\rho}\left((b-\rho)\lambda^{-\frac{\rho}{b-\rho}}-b\right)\right)-1}{a^{\frac{\rho}{b-\rho}}-1}-1$$

$$F^* =$$

$$\frac{a^{\frac{\rho}{b-\rho}}\left[1+\frac{1}{\lambda-1}\left(-(a-\lambda)-\frac{a-1}{\rho}\left((b-\rho)\lambda^{-\frac{\rho}{b-\rho}}-b\right)\right)-1\right]-1}{a^{\frac{\rho}{b-\rho}}-1}-1=$$

$$\frac{a^{\frac{\rho}{b-\rho}}\left[1+\frac{1}{\lambda-1}\left(-(a-\lambda)-\frac{a-1}{\rho}\left((b-\rho)\lambda^{-\frac{\rho}{b-\rho}}-b\right)-\lambda+1\right)\right]-1}{a^{\frac{\rho}{b-\rho}}-1}-1=$$

$$\frac{a^{\frac{\rho}{b-\rho}}\left[1+\frac{1}{\lambda-1}\left(-(a-1)-\frac{a-1}{\rho}\left((b-\rho)\lambda^{-\frac{\rho}{b-\rho}}-b\right)\right)\right]-1}{a^{\frac{\rho}{b-\rho}}-1}-1=$$

$$\frac{a^{\frac{\rho}{b-\rho}}\left[1+\frac{a-1}{\lambda-1}\left(-\frac{1}{\rho}\left((b-\rho)\lambda^{-\frac{\rho}{b-\rho}}-b+\rho\right)\right)\right]-1}{a^{\frac{\rho}{b-\rho}}-1}-1=$$

$$F^* =$$

$$\frac{a^{\frac{\rho}{b-\rho}}\left[1+\frac{a-1}{\lambda-1}\frac{b-\rho}{\rho}\left(1-\lambda^{-\frac{\rho}{b-\rho}}\right)\right]-1}{a^{\frac{\rho}{b-\rho}}-1}-1=$$

$$\frac{a^{\frac{\rho}{b-\rho}}}{a^{\frac{\rho}{b-\rho}}-1}\,\frac{\lambda^{\frac{\rho}{b-\rho}}-1}{\lambda^{\frac{\rho}{b-\rho}}}\,\frac{a-1}{\lambda-1}\,\frac{b-\rho}{\rho}$$

which is strictly positive.

References

Aghion, Philippe and Peter Howitt [1992], "A Model of Growth Through Creative Destruction," *Econometrica*, **60**, 323-361.

Boldrin, Michele and David K. Levine [2001], "Factor Saving Innovation, "*Journal of Economic Theory*, **105**: 18-41.

Boldrin, Michele and David K. Levine [2004], "Rent Seeking and Innovation," *Journal of Monetary Economics*, **51**: 127-160.

http://www.businessweek.com/magazine/content/02_45/b3807117.htm

Grossman, Gene and Elhanan Helpman [1991], "Quality Ladders in the Theory of Growth," *The Review of Economic Studies*, **58**: 43-61.

Hannan, Timothy H. and John M. McDowell [1987], "Rival Precedence and the Dynamics of Technology Adoption: An Empirical Analysis**,"** *Economica*, New Series, **54**: 155-171

Hirshleifer, Jack [1971], "The Private and Social Value of Information and the Reward to Inventive Activity," *The American Economic Review*, **61**: 561-574.

Irwin, Douglas A. and Peter J. Klenow [1994], "Learning-by-Doing Spillovers in the Semiconductor Industry," *The Journal of Political Economy*, **102**: 1200-1227.

Manuelli, Rody E. and Ananth Seshadri [2003], "Frictionless Technology Diffusion: The Case of Tractors," NBER Working Paper No. W9604.

Park, Sangin [2002], "An Empirical Evaluation of the 1986 Semiconductor Trade Arrangement," mimeo SUNY Stony Brook.

Romer, Paul M. [1990], "Endogenous Technological Change," *Journal of Political Economy*, **98**: S78-S102.

Rose, Nancy L. and Paul L. Joskow [1990], "The Diffusion of New Technologies: Evidence from the Electric Utility Industry**,"***RAND Journal of Economics*, **21**: 354-373.

Sarkar, Jayati [1998], "Technological Diffusion: Alternative Theories and Historical Evidence," *Journal of Economic Surveys*, 12: 131 -176.