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Fixed Income Mathematics



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Abbreviations

<i>FV</i>	Future value. It expresses the value of a cash flow at a given future date.
<i>PV</i>	Present value. It expresses the value of a future cash flow as of the date of valuation.
<i>r</i>	Interest rate. It denotes the annual interest rate, unless otherwise indicated.
<i>n</i>	Number of years.
<i>days</i>	Number of days.
<i>d</i>	Discount rate.
<i>yr</i>	Number of days in a year.
<i>m</i>	Number of interest payment periods (within a year).
<i>P</i>	Price. This term is usually used interchangeably with present value, as the present value of future cash flows is considered the fair price of an asset. The designation P_0 places more emphasis on present value. In this sense, prices at other dates can be also denoted, i.e. P_1, P_2, \dots, P_n indicate prices 1, 2, ... n periods later.
<i>CF</i>	Cash flow. The designation CF_0 expresses a present value, often the initial investment. Future cash flows are denoted CF_1, CF_2, \dots, CF_n , respectively.
<i>M</i>	Maturity value. This can be either the face value of a bond or the expected sales price of a financial instrument, but it can also be the terminal value expected to be realised when a project is completed.
<i>AI</i>	Accrued interest. The pro rata portion of a bond's coupon between two coupon payments.
<i>C</i>	Coupon.
<i>e</i>	Euler's number, the base of the natural logarithm. It is an irrational number with a constant value: 2.718...
<i>T</i>	Time. T_0 denotes the present or the current day. Accordingly, T_1, T_2, \dots, T_n express dates 1, 2, ... n periods later, typically 1, 2... n years later.
<i>NPV</i>	Net present value.
<i>YTM</i>	Yield to maturity.

Δ	Greek delta. Denotes change in the value of a variable.
DF	Discount factor.
DUR	Duration.
$CONV$	Convexity.
S	Spot rate or zero-coupon rate.
F	Forward rate.
w	Weight.

1 Introduction

Better an egg today than a hen tomorrow, as the saying goes, and this could also be the tenet of modern finance. This is because two cash flows received at different times have different values. 100 forints today is not worth the same as 100 forints one year from now. In fact, it is possible that 100 forints today is worth more than 105 forints will be worth one year from now. This is the time value of money, or more broadly speaking, its consumption value, and it depends on the interest rate for the given period or, in layman's terms, the yield that can be achieved in a certain period.

Financial instruments and receivables can also be categorised based on whether the time and size of the cash flow they provide is known in advance, and whether we have certainty with respect to these two parameters. In the case of a "casco" insurance, which is a non-life insurance product, we do not know the extent of the damage, or, accordingly, the amount paid by the insurer, or the time when the damage may occur. Therefore, neither the time nor the size of the cash flow is known, and both parameters are uncertain.

In the case of life insurance, for example, especially so-called endowment insurance, the amount of the sum insured is known, but we do not know when the event will take place. In the case of shares, dividends are paid at a predetermined time, in Europe typically once a year, but their amount is uncertain, since we cannot know how profitable the given company will be in a certain year, i.e. the amount the company is able to pay to its owners.

Chart 1
Categories of financial instruments by the time and size of their cash flow

		Time	
		<i>Certain</i>	<i>Uncertain</i>
Size	<i>Certain</i>	bonds	life insurance
	<i>Uncertain</i>	shares	non-life insurance

Finally, there are financial instruments in the case of which both the time and the size of the cash flow are known beforehand. These are known collectively as fixed-income instruments. This category is, of course, not completely homogenous: for example, in the case of a variable-rate bond one might ask why its cash flow is fixed when it has a variable interest rate, not to mention even more complex instruments containing a so-called optionality, such as US mortgage bonds or callable bonds. The common denominator in these instruments is that the rule based on which their cash flows are determined does not change during the “life” of the bond. There is no uncertainty typical in the other three categories, which makes the valuation and pricing of these instruments easier. This handbook focuses on the valuation of fixed-income instruments. In addition to calculating the price, our aim is to understand the factors influencing it as well as the impact of these factors, i.e. the dynamics of price changes.

2 Bonds

Bonds are debt securities. The issuer of a bond undertakes to pay the holder of the bond the predetermined interest or other commissions of the amount (principal) stipulated in the contract as well as the amount itself at a predetermined time, i.e. at maturity. The time until maturity is called the term to maturity (term) or the tenor of the bond. Bonds can be categorised according to several criteria, and we will present their various forms by detailing their categories.

The notional value of a security is called its notional or face value. Face value means the nominal amount payable as principal, which typically happens at maturity. The nominal interest on the bond obligation paid at predetermined times is called the coupon. The nominal value of the interest payable is determined by the coupon rate relative to face value. In the following, we examine the characteristics of bonds in order to understand the various structures that can be formed by the combination of these parameters, and to lay the basis for bond valuation by introducing the necessary concepts.

In the valuation of a bond, i.e. when determining its price, its cash flow stream can be established based on the face value, the maturity, the coupon rate, etc. This cash flow stream is often charted on a so-called timeline, which visually represents the amounts paid by the bond in each period. This cash flow forms the basis for the bond valuation.

Bonds are issued on the so-called primary market: lenders receive the debt securities and borrowers obtain the funding necessary for financing their activities. Later, during the bonds' term, they are traded on the so-called secondary market: at this time the securities are traded among bondholders. The most recently issued bonds are called on-the-run papers, while older series are called off-the-run papers (e.g. 10-year bonds with a residual maturity of 2 years).

2.1 Interest payment

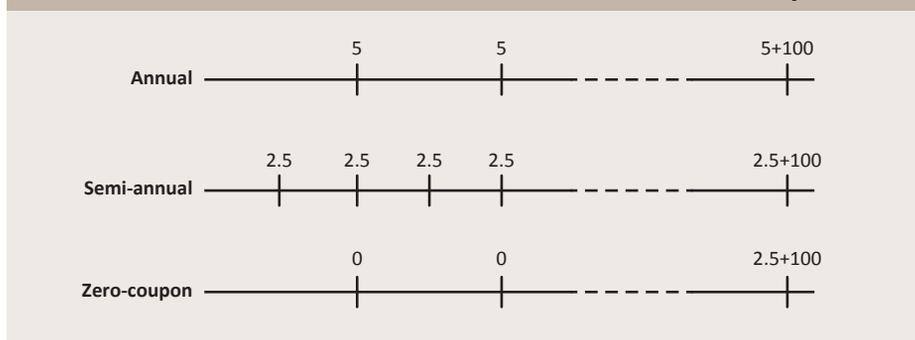
Fixed interest rate

The interest or coupon on a bond is the nominal interest on the bond obligation paid at predetermined times. Bonds can be issued with a wide range of interest rates, but the two basic types are fixed-rate and variable-rate bonds.

As the name suggests, a fixed-rate bond means that its interest does not change during maturity. Bonds pay the nominal interest rate for the given period at the intervals determined by the interest payment frequency. Usually, in the case of fixed-coupon bonds, the interest payment frequency is annual or semi-annual. In the case of annual interest payments, the interest for the whole year is received on a predetermined day, while in the case of semi-annual frequency, the bond pays half of the amount semi-annually.

For example, in the case of a 5 per cent interest rate and a face value of HUF 100, the annual interest is HUF 5, but while in the case of an annual coupon period the only annual cash flow is HUF 5, in the case of a semi-annual coupon period the bond pays HUF 2.5 both at mid-year and at year end. A special case of the fixed interest rate is when the interest is 0. This is called a zero-coupon bond, the only cash flow of which is the principal repaid at maturity.

Chart 2
Cash flows of annual fixed-rate, semi-annual fixed-rate and zero-coupon bonds



Variable or floating interest rate

By contrast, a variable or floating interest rate changes over time. Usually, the interest rate is tied to an interbank rate used as a reference rate. The payment frequency of a floating rate bond typically follows the maturity of the reference rate. For example, if the reference rate has a term of 3 months, then the rate is fixed every three months, and the bond pays quarterly interest every three months.

Perhaps the best-known and most important reference rate in the world is the LIBOR, the London Interbank Offered Rate. The so-called fixing, during which 16 banks quote the loan interest rate charged to each other on the interbank market, is coordinated by the British Bankers' Association and occurs daily. The LIBOR is quoted for maturities between 1 day and 1 year and for the 10 most important currencies. Its significance is due to the fact that a large amount of loans (e.g. corporate loans, mortgage loans), bonds and derivative transactions (e.g. interest rate futures, swaps) are priced based on the LIBOR rates.

Similar to LIBOR, interest rates for different maturities are fixed on a daily basis on the markets for most currencies. The interest rate fixed on the Hungarian interbank forint market is the Budapest Interbank Offered Rate (BUBOR).

These reference rates can be used to develop various interest rate structures. Coupon rates are based on the reference rate, to which a premium expressing the risk factor of the borrower is usually added. For example, the coupon rate of a floating-rate corporate bond could be expressed as the 3-month LIBOR rate + 1 per cent premium. In such a scenario, the interest payment frequency of the bond is quarterly, as the reference rate has a term of 3 months, i.e. one quarter of a year. In the first coupon period, the bond pays interest based on the 3-month LIBOR rate fixed on the issue date. If the 3-month LIBOR rate on that day was 2.5 per cent, then 3 months later the bond pays the pro rata portion of $2.5+1=3.5$ per cent for the quarter ($3.5/4$ of the HUF 100 face value, which is HUF 0.875). The interest paid for the next quarter will be the 3-month LIBOR rate fixed on the day of the coupon payment. As the 3-month LIBOR changes during the tenor of the bond, the interest amount paid by the bond changes (floats) as well.

When determining the exact amount of interest, the so-called day count convention used for calculating the number of days between the two dates is an important factor. There are several methods for this, only one of which is the calculation using the actual number of days between the two dates. In the above example, we simply divided the annual interest by four, although a given quarter is not necessarily precisely one quarter of a year, as there are 89, 90, 91 and 92-day quarters.

Bonds with a special interest rate

In addition to the above-mentioned examples, there are bonds with special, alternative rates, for example a “step-up” interest rate. This means that the interest rate is fixed in advance, but its amount is not constant. For example, it might be 2 per cent in the first year, 2.20 per cent in the second, 2.4 per cent in the third and so on.

A special case of a variable interest rate is when the rate has a predetermined maximum (cap) or minimum (floor). In addition, there are so-called inverse-interest bonds, the interest rate of which changes in the opposite direction to the shift in the reference rate (e.g. 10% - LIBOR).

2.2 Issuers

Bonds can also be categorised by issuers. The issuer can be, for example, a government, a company, a municipality, a supranational institution or a government agency.

Government or sovereign issuers are central players in all bond markets. This is because in its own currency, the state is regarded as a so-called credit risk-free issuer, acting as a point of reference and playing a benchmark role in the valuation of other issuers' bonds. What guarantees that the cash flow determined during the valuation of the bond is in fact received? In the case of the sovereign issuer, it is the fact that the state can use its taxing rights to obtain the money necessary for the repayment of the principal of the government bonds and to meet the obligations stipulated in the bond contract. Other issuers cannot do that, since they do not possess the necessary rights or power.

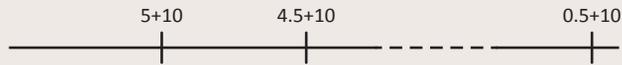
Another aspect why the sovereign bond market plays a key role in the capital market of all countries is market liquidity. In most countries, the state is the largest issuer or at least a prominent one. Due to the need for financing the current expenditure of the budget and for refinancing government debt, the state enters the market with a issuance plan published at regular intervals, usually in advance. In addition to the financing requirement, the issuance plan takes into account the needs of bond investors for certain maturities, and usually its stated aims include market building, i.e. providing a liquid market with large issue volumes and huge trading volumes along the whole maturity spectrum (yield curve), which can act as a true point of reference for the valuation of smaller, riskier bonds.

2.3 Repayment methods

The obligations laid down in the bond contract concern the interest payment and the repayment of the principal. The maturity period of the bonds lasts until the whole principal is repaid. Perhaps one of the easiest ways is when the bond repays the whole face value at maturity as a lump sum (bullet structure).

An alternative to this is when the partial repayment of the principal, and thereby the reduction of the face value, starts during the maturity period in accordance with some predefined rule. This is the so-called amortising structure. The principal may be amortised during the maturity period at a steady pace. One such case is when a 10-year bond repays 10 per cent of the principal each year, or after a grace period, e.g. 5 years, repays 20 per cent annually in the next 5 years. In the case of an amortising principal, the coupon rate of the bond does not change, but as the rate is a portion of the face value, the interest amount to be paid decreases each year. For example if the annual amortisation is 10 per cent, in the second year the interest only has to be calculated on 90 per cent of the original face value. In the case of a 5-per cent coupon rate and an original face value of HUF 100, the interest payable in the second year is only HUF 4.5.

Chart 3
Cash flow of amortising bonds



The amortisation of the principal may also be non-linear. For example, in the case of so-called asset-backed securities, the principal repayment may be coupled with the principal repayment of underlying assets (e.g. mortgage loans), and therefore when the bond's cash flow is sought to be determined for valuation, further assumptions have to be made as regards the pace of the principal repayment. In such a scenario, the forecast of the bond's cash flow is performed on the basis of models.

2.4 Maturities

The last cash flow of the bond is due at the bond's maturity. This basically means the end of the bond's "life". There are a wide range of structures with respect to maturity as well.

The simplest case is fixed maturity when maturity is linked to a predetermined date. By contrast, a special case is when the bond does not have a maturity, and the payment of interest basically continues in perpetuity.

Finally, in order to help understand the relationships discussed later, it must be noted that the bond contract may contain elements which make the maturity of the bond uncertain. For example, a bond can be callable. In this case, the right of redemption lies with the issuer of the bond, providing them with an option (optionality) to call, i.e. to buy back the bond before final maturity, at a predetermined date or dates and at a predetermined price. Similarly, a bond may be puttable. In this case, the bondholder (lender) has the right to give back (sell) the bond at a predetermined price and at a predetermined date or dates to the issuer.

Convertible bonds are a very special case. With these instruments, the bondholder has the right to convert the bond to a certain number of shares if specific conditions are met. In such a scenario, the bond ceases to exist, and from the perspective of the company the debt financing becomes equity financing without maturity, while the bondholder is transformed from being a lender into being a stockholder of the company.

2.5 Collateral

As regards collateral, bonds can be either secured or unsecured. In the case of unsecured securities, the bond is backed by the general obligation or pledge of the issuer.

In the case of secured issues, however, the cash flow is guaranteed by predefined collateral, from which the issuer can fulfil its bond obligations. Such collateral can be a project (e.g. a toll motorway) or a portfolio (e.g. mortgage portfolio).

3 Interest calculation

3.1 Simple interest calculation

When calculating simple interest, the interest received on the principal is not assumed to be capitalised or reinvested, i.e. the interest does not bear interest anymore. This interest calculation method is typically used for short-term financial market instruments.

Such instruments include interest-bearing time deposits with a fixed interest rate and maturity. The maturity period may be between 1 day and several years, but is usually less than 1 year. Interest is paid at maturity.

$$FV = PV \cdot (1 + r \cdot n)$$

Example 1: Interest on bank deposits

How much will our savings be worth if an initial amount of HUF 100 is deposited with a bank for 1 year and the deposit rate is 5%?

$$FV = 100 \cdot (1 + 0.05 \cdot 1) = 105$$

How much will our savings be worth if we deposit our money for 2 years?

$$FV = 100 \cdot (1 + 0.05 \cdot 2) = 110$$

Of course, the duration of the investment can also be a truncated period, instead of a full year. In the case of maturities of less than a year, the interest is prorated based on the duration of the investment relative to the number of days in the year. Different markets use different conventions for determining the length of the partial period and the full year.¹

Since it allowed for easier calculation, the most widespread day count convention used to be the so-called 30/360 convention. In this, every year is

¹ For further information, see: https://en.wikipedia.org/wiki/Day_count_convention

treated as having 360 days and every month is considered to have 30 days. There were obvious reasons for this simplification before the IT revolution, since it is much easier to mentally compute the length of the full period and the pro rata interest in this manner. For example in the case of a quarterly interest payment period, 90 days is compared to the full length of a 360-day year, which gives exactly one-fourth.

Today, the most widespread financial market convention is the so-called ACT/360 day count method. ACT is derived from the word “actual” and denotes the actual number of days, while the year is considered to have 360 days.

$$FV = PV \cdot \left[1 + r \cdot \left(\frac{\text{days}}{\text{yr}} \right) \right]$$

Example 2: Interest on a short-term bank deposit

How much will our savings be worth if an initial amount of HUF 100 is deposited with a bank between 11 May 2015 and 11 August 2015 and the deposit rate is 5%?

$$FV = 100 \cdot \left[1 + 0.05 \cdot \left(\frac{92}{360} \right) \right] = 101.278$$

Note how much easier it is to mentally compute one-fourth of the annual interest of HUF 5 received for the 3-month deposit – the figures from Example 2 – using the 30/360 method. In reality, a quarter may have 89, 90, 91 or 92 days, depending on the months it includes and whether it falls during a leap year.

3.2 Compound interest calculation

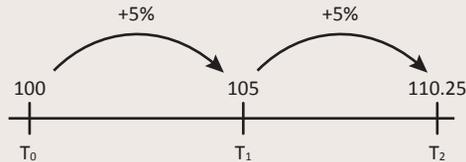
When calculating compound interest, the cash flow of the given instrument is assumed to be reinvested with the same interest.

$$FV = PV \cdot (1+r)^n$$

Example 3: Compound interest

How much will our savings be worth if an initial amount of HUF 100 is deposited with a bank for 2 years, the deposit rate is 5%, and we use compound interest?

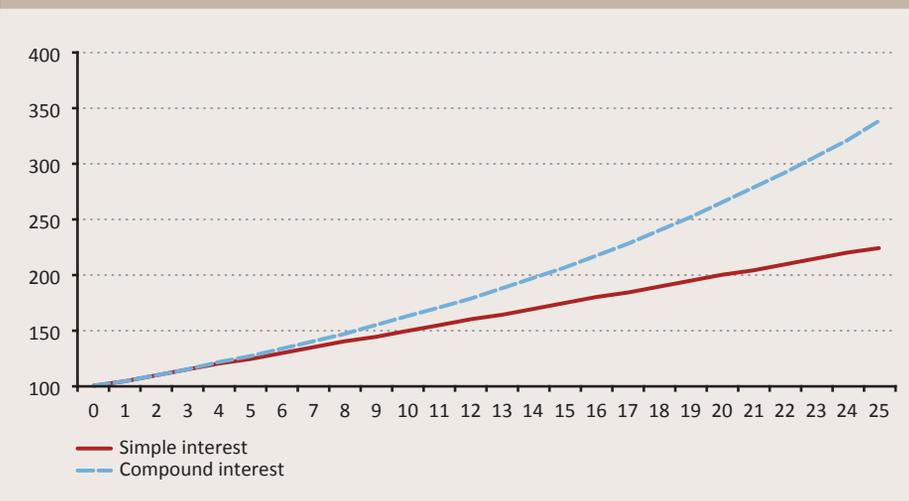
$$FV = 100 \cdot (1 + 0.05)^2 = 110.25$$



We can see that in contrast to simple interest calculation, the HUF 5 interest accrued in the first year is deposited for 1 year in the second period, and interest is earned on that as well.

Chart 4

Impact of simple and compound interest on the value of the investment



The difference expands in proportion to the interest rate and the duration of the investment, in an exponential manner.

Chart 5
Impact of the interest rate on the value of the investment



The interest period can also be different than 1 year. For example, on the US bond market the general market convention is semi-annual interest payment frequency. Using the above example, the 5 per cent interest on HUF 100, or USD 100 on the US markets, is paid in two instalments: USD 2.50 mid-year and another USD 2.50 at the end of the year. Accordingly, the interest obtained at mid-year can be invested in the second half of the year, and thus an additional yield can be realised.

$$FV = PV \cdot \left(1 + \frac{r}{m}\right)^{nm}$$

Example 4: Compound interest with semi-annual interest payments

How much will our savings be worth if an initial amount of HUF 100 is deposited with a bank for 2 years, the deposit rate is 5%, and interest is paid semi-annually?

$$FV = 100 \cdot \left(1 + \frac{0.05}{2}\right)^{2 \cdot 2} = 110.38$$

Of course, the frequency of interest payments can be increased further. If in the above example quarterly interest payments are assumed, the value of

our savings increases to HUF 110.45 by the end of the second year. Interest payments could be made daily, and in such a case the interest earned on the first day could be reinvested on the second. Continuing along these lines, we arrive at the concept of continuous compounding. In this case, we assume that a small amount of interest is paid every moment (continuously). Of course, the actual payment of the interest amount itself is not feasible in everyday life, but the concept of continuous compounding not only has theoretical significance, as it is used in the pricing of several financial products (e.g. options).

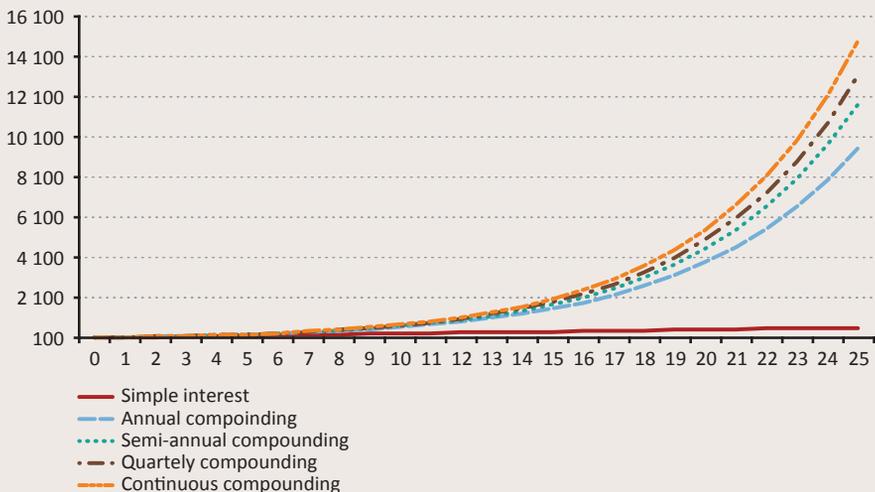
$$FV = PV \cdot e^{nr}$$

Example 5: Continuous compounding

How much will our savings be worth if an initial amount of HUF 100 is deposited with a bank for 2 years, the deposit rate is 5%, and continuous compounding is assumed?

$$FV = 100 \cdot e^{0.05 \cdot 2} = 110.517$$

Chart 6
Impact of interest payment frequency on the value of the investment



²This is because $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right) = e$.

4 Present value calculation

4.1 Calculating present value

The value of two cash flows received at two different times is not necessarily the same. An amount received in the future cannot be invested today, and therefore the yields potentially accrued until the future date are not realised. Thus, in the case of a future cash flow, we have to calculate the yield that could be earned with it on the invested capital, should it be available to us today. The value of a future cash flow calculated for T_0 , i.e. today, is the present value. Present value calculation itself is also called discounting.

$$PV = \frac{FV}{(1+r)^n}$$

Example 6: Present value calculation

What is the present value of a cash flow of HUF 100 received in 1 year if the interest rate is 5%?

$$PV = \frac{100}{(1+0.05)^1} = 95.24$$

What is the present value of a cash flow of HUF 100 received in 2 years if the interest rate is 5%?

$$PV = \frac{100}{(1+0.05)^2} = 90.703$$

Using the same interest rate, the further away a cash flow is, the smaller its present value, and over a given period, the higher the interest rate, the smaller the present value.

Chart 7
Present value of a cash flow of HUF 100 depending on the duration and the interest rate

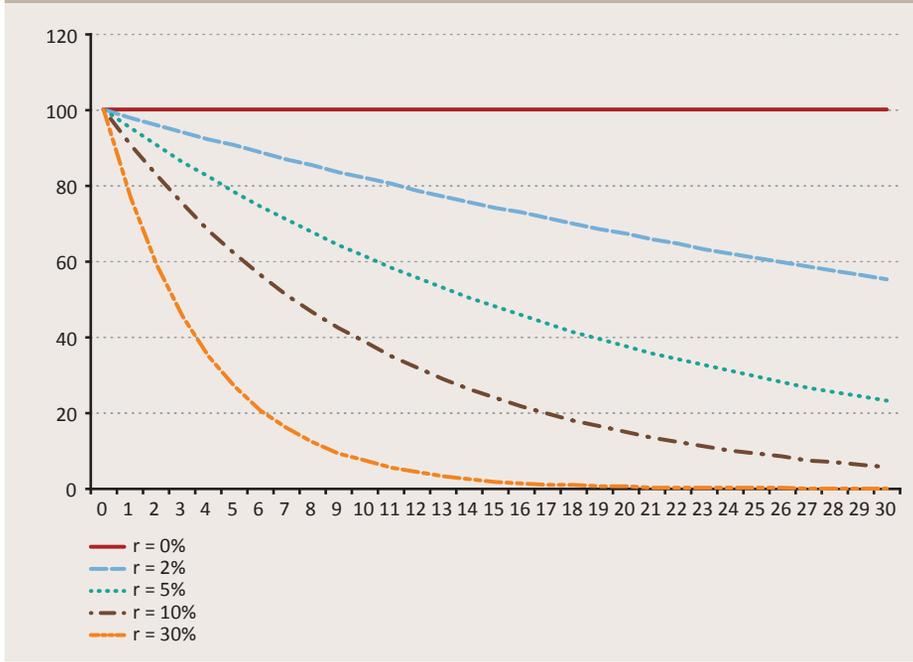


Chart 7 shows the above relationship between the present value of a future cash flow of HUF 100 and the time until the cash flow is received as well as the interest rate used for discounting. Obviously, when the interest rate is 0, the present value of a future cash flow equals its nominal value.

In the case of a short-term cash flow less than 1 year away, simple interest is used for calculating the present value.

$$PV = \frac{FV}{\left(1 + r \cdot \frac{\text{days}}{\text{yr}}\right)}$$

Example 7: Present value calculation in the case of short-term cash flows

What is the value of a cash flow of HUF 100 received in 3 months if the interest rate is 5%?

$$PV = \frac{100}{\left(1 + 0.05 \cdot \frac{90}{360}\right)} = 98.765$$

The factor linking the present value (*PV*) and the future value (*FV*) in the formula for the present value calculation is called the discount factor. This can also be seen as the present value of the cash flow of HUF 1 due at a certain future time. Naturally, the present value of any cash flow can be calculated by multiplying it by this number.

$$DF = \frac{1}{(1+r)^n}$$

Table 1

Discount factor – Present value of HUF 1

		Discount rate														
		1%	2%	3%	4%	5%	6%	7%	8%	9%	10%	11%	12%	13%	14%	15%
Years	1	0.99	0.98	0.97	0.96	0.95	0.94	0.93	0.93	0.92	0.91	0.90	0.89	0.88	0.88	0.87
	2	0.98	0.96	0.94	0.92	0.91	0.89	0.87	0.86	0.84	0.83	0.81	0.80	0.78	0.77	0.76
	3	0.97	0.94	0.92	0.89	0.86	0.84	0.82	0.79	0.77	0.75	0.73	0.71	0.69	0.67	0.66
	4	0.96	0.92	0.89	0.85	0.82	0.79	0.76	0.74	0.71	0.68	0.66	0.64	0.61	0.59	0.57
	5	0.95	0.91	0.86	0.82	0.78	0.75	0.71	0.68	0.65	0.62	0.59	0.57	0.54	0.52	0.50
	6	0.94	0.89	0.84	0.79	0.75	0.70	0.67	0.63	0.60	0.56	0.53	0.51	0.48	0.46	0.43
	7	0.93	0.87	0.81	0.76	0.71	0.67	0.62	0.58	0.55	0.51	0.48	0.45	0.43	0.40	0.38
	8	0.92	0.85	0.79	0.73	0.68	0.63	0.58	0.54	0.50	0.47	0.43	0.40	0.38	0.35	0.33
	9	0.91	0.84	0.77	0.70	0.64	0.59	0.54	0.50	0.46	0.42	0.39	0.36	0.33	0.31	0.28
	10	0.91	0.82	0.74	0.68	0.61	0.56	0.51	0.46	0.42	0.39	0.35	0.32	0.29	0.27	0.25

For example, the present value of HUF 1 due in 1 year with an interest rate of 1 per cent is HUF 0.99, while the present value of HUF 1 due in 10 years with an interest rate of 15 per cent is only HUF 0.25. The cash flow's temporal shift towards the future and a rise in the interest rate reduces the present value at an accelerating pace.

4.2 Discount calculation

There are instruments in the case of which a so-called discount rate is quoted instead of an interest rate. In their case, the present value is calculated by subtracting the discount from the future value.³

$$PV = FV \cdot (1 - d)$$

Since these are typically short-term financial market instruments (e.g. US discount Treasury bills or T-bills), the discount is prorated for the maturity using the simple interest calculation method.

$$PV = FV \cdot \left(1 - d \frac{\text{days}}{\text{yr}}\right)$$

Example 8: Discounting

A US Treasury bill with a face value of USD 100 is issued at a discount rate of 2.53 per cent. Residual maturity is 83 days. What is the price of the paper?

$$PV = 100 \cdot \left(1 - 0.0253 \cdot \frac{83}{360}\right) = 99.417$$

4.3 Bond price calculation

For calculating the price of bond-type instruments, the present value calculation described above is used. Actually, in the case of any instrument, the future cash flows provided by it are taken into account during its valuation, even if the value of an instrument is not determined solely by these cash flows. In the case of certain assets (e.g. works of art), there is no cash flow that could be discounted, yet the asset has a value: it is traded from time to time, and not at a price of 0.

³ Similar to what we encounter daily in retail sales.

An asset may have a value in use or aesthetic value, but in the financial markets the only factors that count are the time and size of the expected cash flow, and the interest rate used for discounting the given cash flow. Among financial instruments, the valuation of fixed-income instruments is the easiest, since the cash flow is known in advance or at least it is controlled by rules known in advance.⁴

The simplest instrument is the so-called zero-coupon bond, which only has one cash flow. In the case of short-term securities, simple interest is assumed during discounting, while in the case of securities with maturities of over one year, compound interest is assumed.

Example 9: Price of zero-coupon bonds

What is the price of discount Treasury bill with a face value of HUF 100 maturing in 92 days if the discount rate is 2.15%?

$$P_0 = \frac{100}{\left(1 + 0.0215 \cdot \frac{92}{360}\right)} = 99.4536$$

What happens if more than one cash flow has to be discounted? Bonds are so-called interest-bearing instruments, and interest is paid at regular intervals before maturity, while the principal is repaid at maturity.

⁴ In order to facilitate the understanding of the principles and process of price calculation, substantial simplifications are made. First, certainty is assumed with respect to the cash flow to be received. It is certain that a bond with a face value of HUF 100 and a coupon of 5 per cent should pay a coupon of HUF 5 annually until maturity, but even sovereign issuers, which are considered safe from a risk perspective, pose a default risk, i.e. in practice we cannot be sure that we will receive the HUF 5 coupon annually and the HUF 100 face value at maturity. The other simplifying assumption used at this point is that the same interest rate (r) is used for discounting all future cash flows.

Chart 8
Cash flow of a 10-year, fixed-rate, 5% annual coupon bond with a face value of HUF 100



This 10-element cash flow stream can be regarded as 10 separate cash flows discounted one by one. Then the present values of all 10 are summed, which gives us the present value of the total cash flow.

The most general formula possible used for the calculation is as follows:

$$P_0 = \sum_{n=1}^N \frac{CF_n}{(1+r)^n}$$

In the case of a bond, coupon payments and principal repayment can be distinguished. We can see that in the case of a zero-coupon bond, the value of coupon payments is zero, and there is nothing else other than the repayment of the principal at maturity.

$$P_0 = \frac{CF_1}{(1+r)} + \frac{CF_2}{(1+r)^2} + \frac{CF_3}{(1+r)^3} + \dots + \frac{CF_N}{(1+r)^N} + \frac{M}{(1+r)^N} = \sum_{n=1}^N \frac{CF_n}{(1+r)^n} + \frac{M}{(1+r)^N}$$

In addition, in the case of fixed-rate bonds, the coupon due annually does not change, therefore:

$$P_0 = \frac{C}{(1+r)} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \dots + \frac{C}{(1+r)^N} + \frac{M}{(1+r)^N}.$$

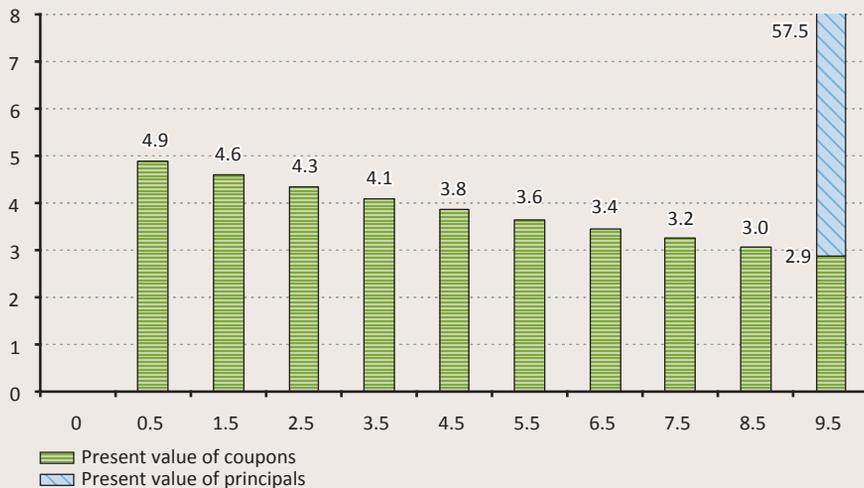
Example 10: Price of fixed-coupon bonds

What is the price of a 10-year, fixed-rate, 5-per cent annual coupon bond with a face value of HUF 100 and an interest rate of 6%?

$$P_0 = \frac{5}{(1+0.06)} + \frac{5}{(1+0.06)^2} + \frac{5}{(1+0.06)^3} + \dots + \frac{5}{(1+0.06)^{10}} + \frac{100}{(1+0.06)^{10}} = 4.7 + 4.4 + 4.2 + 4.0 + 3.7 + 3.5 + 3.3 + 3.1 + 3.0 + 2.8 + 55.8 = 92.6$$

Chart 9

Present value of each cash flow when the interest rate is 6%



5 Yield calculation

The yield is the investor's profit. This is the actual profit – preferably not loss – realised on an investment. In this sense, there may be no direct relationship between the interest and the yield. The interest amount is calculated relative to the face value (coupon), and the final profit from an investment does not necessarily equal this.

When considering investments, we must decide whether it is worth initiating them. For example whether it is worth it to buy a machine in order to sell the equipment manufactured with it and make a profit. The financial plan prepared for the valuation of the project takes into account the expected net revenue, i.e. the cash flow stream resulting from the project, the value of the initial investment and the financing cost of the project. The latter is the interest rate used for discounting the cash flow.

5.1. Net present value

With all investments, the present value of the cash flow stream derived from the asset or project is basically compared to the initial value of the investment. The difference between the two is the net present value. If the net present value is positive, i.e. greater than zero, then the investment should be made. If it is negative, then it should not be made.

$$NPV = CF_0 + \sum_{n=1}^N \frac{CF_n}{(1+r)^n}$$

Example 11: Net present value of investments

The following information is available during the valuation of an investment. The value of the machine to be purchased is HUF 250. After deducting production and sales costs, the sales of the goods produced with the machine provide an after-tax profit of HUF 80, 90 and 120, respectively, in the next 3 years. Should the investment be initiated if the discount rate is 6 per cent?

$$NPV = (-250) + \frac{80}{(1+0.06)} + \frac{100}{(1+0.06)^2} + \frac{120}{(1+0.06)^3} = 15.2$$

As the net present value is greater than zero ($NPV > 0$), the answer is yes, the investment is worth making.

5.2 Internal rate of return

Analysis of the above formula shows that as the interest rate used for discounting increases, the present value of the cash flows and thus also the total net present value of the project decreases. There is a point where net present value is exactly zero. The corresponding interest rate is called the internal rate of return (*IRR*).

$$NPV = CF_0 + \sum_{n=1}^N \frac{CF_n}{(1+IRR)^n} = 0$$

The internal rate of return can be considered a sort of break-even rate that provides further information for the valuation of an investment, in addition to net present value. This is because net present value is “only” a nominal value that does show much in itself. It is hard to judge the quality of the investment indicated by the HUF 15.2 calculated in the above example, i.e. whether this is a “good” result, whether it is much or little.

Example 12: Internal rate of return of investments

The following information is available during the valuation of an investment. The value of the machine to be purchased is HUF 250. After deducting production and sales costs, the sales of the goods produced with the machine provide an annual after-tax profit of HUF 80, 90 and 120, respectively, in the next 3 years. What is the internal rate of return of the project?

$$0 = (-250) + \frac{80}{(1+IRR)} + \frac{100}{(1+IRR)^2} + \frac{120}{(1+IRR)^3}$$
$$IRR = 0.0905$$

Therefore the rate of return for the project is 9.05%.

Comparison of the internal rate of return to the capital cost of the project provides further assistance in making the decision. Just like in the case of a positive net present value, the project is worth launching if its internal rate of return is higher than the capital cost (in this example 9.05 per cent compared to 6 per cent). In addition to the sign of the difference (whether it is positive or negative), its size is also interesting, as the amount by which the rate of return exceeds the expected capital cost is relevant.⁵

5.3 Yield to maturity

The valuation of financial investments is supported by a similar framework. There is an observed market price for which an asset can be bought, and there has to be an assumption about the cash flows generated by the asset as well as the capital cost (opportunity cost). The estimated price of the asset can be derived from the estimation of the cash flow and the interest rate used for discounting, and by comparing that to the market price we can decide whether it is worth investing in the given asset. In other words: we can

⁵ The qualitative assessment of the difference is beyond the scope of the present handbook. Of course, the valuation is not independent from the context (e.g. market environment), i.e. we have to know what assumptions were made during the estimation of the cash flows or the capital cost and why.

calculate the internal rate of return of the asset, which then can be compared to the yield of alternative investment instruments.

In the case of bonds, the internal rate of return used in the above formula is called yield to maturity (YTM).

$$P_0 = \sum_{n=1}^N \frac{CF_n}{(1+YTM)^n} + \frac{M}{(1+YTM)^N} = \frac{C}{(1+YTM)} + \frac{C}{(1+YTM)^2} + \dots + \frac{C}{(1+YTM)^N} + \frac{M}{(1+YTM)^N}$$

When regarding the price of the bond as the initial investment (CF_0), the net present value of the investment is zero at the yield to maturity. Actually, the “interest rate” (r) that we have used in the formulas until now equals the yield to maturity.

Example 13: Yield to maturity

What is the yield to maturity of a 10-year, 5-per cent annual coupon bond with a face value of HUF 100, the current price of which is HUF 92.64?

$$92.64 = \frac{5}{(1+YTM)} + \frac{5}{(1+YTM)^2} + \dots + \frac{105}{(1+YTM)^{10}}$$

$$YTM = 0,06, \text{ i.e. } 6\%$$

The yield of a bond (the basis for market quotations) refers to the yield to maturity. Nevertheless, it is very important to note that the yield quoted when the bond is purchased does not necessarily equal the return actually realised until the maturity of the investment (total return). From this perspective, the name “yield to maturity” may be misleading, since the actually realised yield may be substantially different from this.⁶

One of the major reasons for this is that in the above formula it was assumed that the cash flows realised during the tenor of the bond are reinvested at the same rate, i.e. at yield to maturity. In reality, however, nothing guarantees that the same market conditions will apply during the maturity period as at the time of the investment.

⁶The actual yield can be different for several reasons, for example because the issuer goes bankrupt, but here we will concentrate on the technical reasons in order to facilitate the understanding of bond pricing mathematics.

5.4 Holding period return

The holding period return (HPR) is the total return realised during the holding period. This is the investor's actual yield.

$$HPR = \frac{V_1 - V_0}{V_0}$$

Example 14: Calculation of holding period return

At the beginning of the year, an investor bought a 2-year bond for HUF 97.50 that pays a face value of HUF 100 at maturity at the end of the second year. Assuming that no other cash flow was derived from the bond, how much was the investor's holding period return?

$$HPR = \frac{100 - 97.50}{97.50} = 0.0256$$

The investor's holding period return was 2.56%.

The above example does not take into account the length of the investment horizon. Had the duration of the investment been 5 years, the holding period return would have been the same. In order to facilitate the comparison of different investments ("apples to apples"), the comparison has to be based on the equivalent of the holding period returns. This equivalent is the annualised holding period return.

$$(1 + HPR) = (1 + r)^n$$

Example 15: Annualised holding period return

Using the figures from the above example, what is the annualised holding period return of the investor?

$$1 + 0.0256 = (1 + r)^2$$

$$r = 0.0127$$

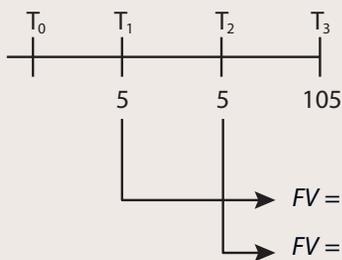
The investor's annual yield was 1.27%.

The 2.56 per cent holding period return was achieved during the two years, which is equivalent to earning a yield of 1.27 per cent each year. The calculation is similar when the holding period return of a non-zero-coupon paper is to be determined. We take the cash flows during the maturity period, calculate the total value of our investment at maturity with the given reinvestment rate, and then this sum is divided by the value of the initial investment. Finally, the holding period return is annualised.

Example 16: Calculation of annualised holding period return

Let us consider a 3-year, 5-per cent annual coupon bond with a face value of HUF 100. What is the annualised holding period return with a reinvestment rate of 2, 5 and 8 per cent, if the initial price was HUF 100?

Year	Cash flow	Future value (CFi)		
		2%	5%	8%
0	-100.00			
1	5.00	5.20	5.51	5.83
2	5.00	5.10	5.25	5.40
3	105.00	105.00	105.00	105.00
	Total FV (Σ)	115.302	115.763	116.232
	HPR	15.30%	15.76%	16.23%
	Annualized yield, r	4.86%	5.00%	5.14%



$$HPR = \frac{115.30 - 100}{100} = 0.1530$$

$$(1 + 0.1530) = (1 + r)^3$$

$$r = 0.0486, \text{ i.e. } 4.86\%$$

It can be seen that the interest generated by our money depends on the different reinvestment rates. Of course, the reinvestment rate can also fluctuate in each year, as the yield environment changes in an economy.

5.5 The yield-price relationship

Price goes up, yield goes down. That is to say, a bond's price and yield are inversely related. As noted earlier: if a higher interest rate is used for discounting a future cash flow, then its present value will be lower (see Chart 7).

Example 17: Developments in bond prices depending on yield

Calculate a 10-year, 5-per cent annual coupon bond's price at yields of 3.00%, 4.00%, 4.98%, 4.99%, 5.00%, 5.01%, 5.02%, 6.00% and 7.00%.

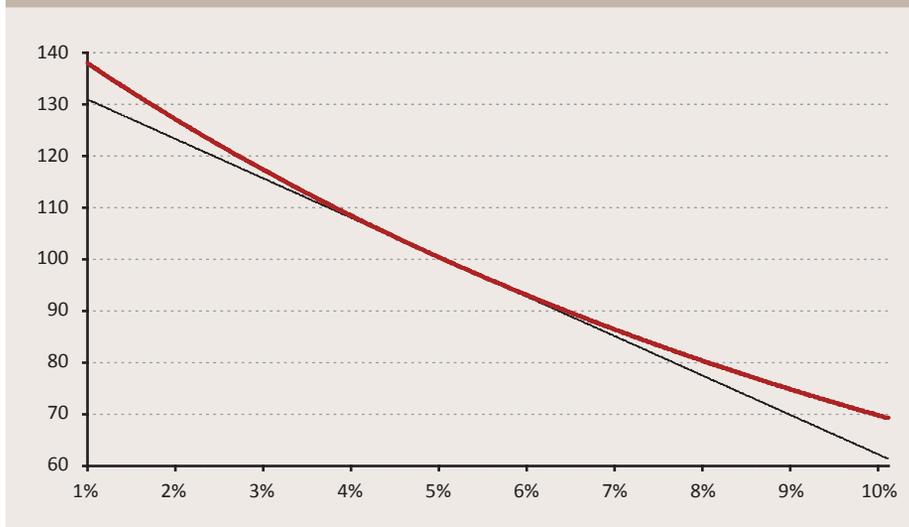
Yield	Price
3.00%	117.06
4.00%	108.11
4.98%	100.15
4.99%	100.08
5.00%	100.00
5.01%	99.92
5.02%	99.85
6.00%	92.64
7.00%	85.95

The above calculation shows not only that the price decreases as the yield increases and that the price rises as the yield drops (i.e. that this is an inverse relationship), but also that the pace of change is uneven. In mathematical terms: it is non-linear.⁷ This means that one unit of yield decrease triggers an ever increasing pace of price increase, while one unit of yield growth triggers an ever slower price drop.

⁷ In the case of a linear function (relationship), the unit change of the explanatory variable triggers the same magnitude of change in the dependent variable. In reality there are few such relationships, but often the relationship between two variables can be well described with a linear function (*linear regression*). As we will see later, in the case of a small change in the yield, the relationship between yield and price can be approximated well with the linear method, but it is vital to understand that in this relationship the extent of change (the first derivative of the function) is not the only thing that matters, as the change in the pace of change (the second derivative of the function) is also important.

For example, if the yield drops from 5 per cent to 4 per cent, the price increases by HUF 8.11 (from 100 to 108.11), and a further 1-percentage point fall in the yield triggers a HUF 8.95 price increase (from 108.11 to 117.06). In the case of a yield increase, the price shrinks less and less: a 1-percentage point change first induces a HUF 7.36 decline and then a HUF 6.69 decline. This means that even the pace of change changes, increasing at a heightening pace in the case of yield increases, and decreasing at a diminishing pace in the case of yield declines.

Chart 10
Relationship between changes in yield and price

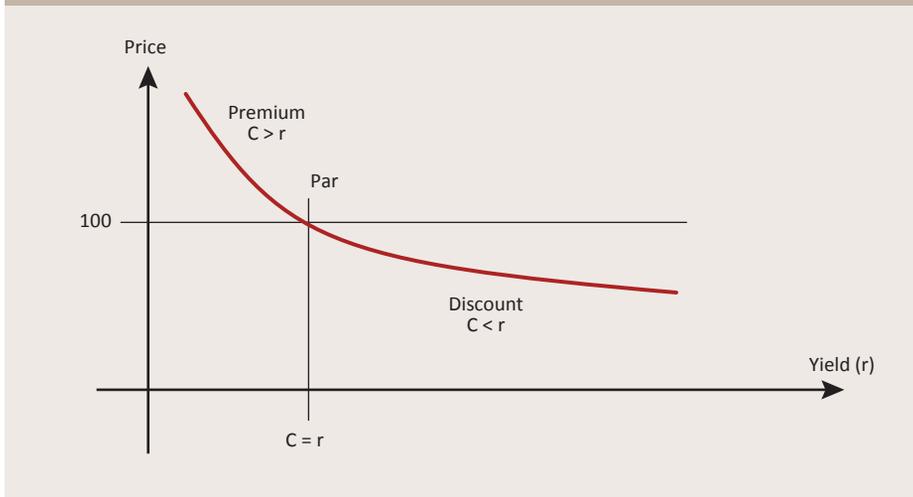


For the purposes of illustration, Chart 10 also contains the imaginary linear line, to demonstrate how the “gap widens”, i.e. how the difference between the assumption of a linear price change and the price calculated using the above formula increases.

One more relationship can be derived from the chart. When the bond’s yield to maturity (i.e. the interest rate used to discount the cash flows) equals the coupon rate of the bond (in this case 5 per cent), the price of the bond is precisely the same as the face value, which in this example is 100. This is called the par price. If the bond’s price is above the face value, then the bond

is trading at a premium, and when it is below the face value, it is trading at a discount.⁸

Chart 11
Discount, par, premium



5.6 “Pull-to-par”

In the case of all assets, the holding period return basically has two sources, and the cash flows from these sources need to be identified during the valuation of the asset. One of the sources may be the income generated by the investment, such as the rental price or the dividend in the case of a property or share, respectively. The other source of return is the change in the price of the given asset.

Accordingly, the holding period return of the bond is made up of two parts as well: 1) the coupon paid at regular intervals and 2) the price change, which

⁸ It should be clarified here that the term “discount bond” can mean two things. First, a bond quoted with a discount rate or discount (e.g. the US discount Treasury bill), in the case of which the present value is calculated by subtracting the discount from the face value rather than dividing by the yield. Second, it could also mean a bond with a price below par, regardless of the fact whether it is a zero-coupon bond or one that pays a coupon at regular intervals.

– when the bond is held until maturity – is the difference between the initial price and the face value paid at maturity.⁹

$$r = C + \Delta P$$

The easiest way to understand the latter is to look at the price development of a zero-coupon bond, where there is no coupon and the yield has only one source.¹⁰

Example 18: Price development of zero-coupon bonds

Calculate the price of a 5-year zero-coupon bond with a residual maturity of 5, 4... etc. years and a 5-per cent yield.

Remaining term to maturity	Price
5	78.35
4	82.27
3	86.38
2	90.70
1	95.24
0	100.00

Even with constant market yields, the price of a bond steadily increases with the passage of time, approaching the face value. This rise is mechanical and is due to the fact that the bond draws ever nearer in time to its only cash flow, and the present value of that cash flow steadily increases. This is the pull-to-par phenomenon, i.e. the approach (pull) of the price towards the face value.

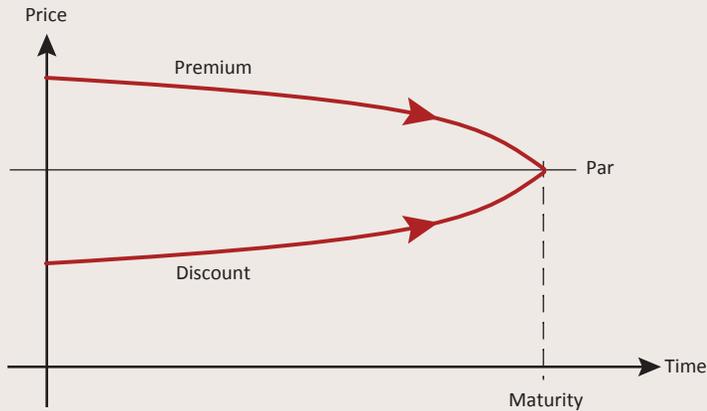
The pull-to-par phenomenon works the same way in the case of coupon bonds: the only difference is that the sign of the contribution from the price's approximation towards the face value differs depending on the relationship between the coupon rate and the yield to maturity. In the case of securities

⁹ At the moment of the cash flow payment, the present value of the payment equals its face value, since when $n=0$ $(1+r)^0=1$. Furthermore, we do not consider here a potential bankruptcy, when issuers cannot fulfil their obligations undertaken in the bond contract.

¹⁰ $C = 0$, therefore $r = \Delta P$.

trading at a discount, the discount portion adds to the yield provided by the coupon ($\Delta P > 0$), while the erosion of the premium “subtracts” from it, i.e. it offsets the coupon yield that is higher than the yield to maturity ($\Delta P < 0$).

Chart 12
The pull-to-par phenomenon



5.7 Accrued interest

The present value formula which we have used until now always calculates the price of the bond for the day of coupon payment, i.e. it is always based on an entire coupon period. In order to reflect reality more closely, the formula must be somewhat modified. In real life, bonds are also traded between two coupon payments, but in such cases no coupon payments are made. Coupon payments only occur at predetermined intervals, e.g. annually, and the coupon amount is transferred to the current owner. Is the last owner entitled to the entire coupon amount, even if they bought the bond on the day prior to the coupon payment? The answer is no: the coupon must be “divided” proportionately between the old owner and the new owner, assuming steady, linear interest accrual.

$$A I = C \cdot \frac{\text{days since the last coupon payment}}{\text{length of coupon period}}$$

The seller receives the interest amount for the days that have passed since the last coupon payment, while the bond buyer only accrues interest after the purchase. This settlement is made at the time of the trade: the buyer basically pays the seller the interest accrued up to that point, and the buyer then receives that amount from the issuer of the bond at the next coupon payment.

In line with market conventions, the quoted price of a bond is the so-called net price (or clean price), i.e. the price without accrued interest. In the course of settlement, the accrued interest is added to this, and this total amount is transferred in exchange for the bond. This is the so-called gross price (or dirty price).

The general formula used in present value calculations provides the present value of the total cash flow for any given day. This already contains accrued interest, i.e. this is basically the gross price.

$$P_0 = \frac{C}{(1+r)^i} + \frac{C}{(1+r)^{1+i}} + \frac{C}{(1+r)^{2+i}} + \dots + \frac{C}{(1+r)^{N-1+i}} + \frac{M}{(1+r)^{N-1+i}}$$

$$i = \frac{\text{days since the last coupon payment}}{\text{length of coupon period}}$$

The present value of a bond's cash flow changes or may change for two reasons from one day to the next: 1) changes in remaining maturity (which is certain), or 2) yield changes, i.e. the interest rate used for discounting (which is not certain). The exact pace and extent of the passage of time is known in advance, and the development of the bond's price (the present value of the cash flows) as a function of the passage of time is easily computable; therefore, the amount of accrued interest for each day can be calculated as well. What we cannot know for certain is how the market environment may change, what the expected yield of the bond will be and thus how the net price of the bond may shift.

Hence, the convention on the bond markets is that price quotation refers to the net price, as that captures every uncertain factor. To facilitate understanding, let us assume that there is a government bond, the yield to maturity and coupon rate of which are equal. The price of the bond is then par.¹¹ If the yield

¹¹ Either the net price is par, or both the net and the gross prices are par at the time of the coupon payment, since then accrued interest is zero.

of this government bond does not change from one day to the next – because fundamental factors such as growth prospects, inflation or the credit risk of a country are constant – then the “clean” price does not change either. It must be noted, however, that if this were not a par bond, the clean price would also change mechanically due to the pull-to-par phenomenon, but we can disregard this effect for now.

Example 19: Developments in gross and net price

Let us chart the developments in the gross and net price of a 2-year, fixed-rate, 5-per cent annual coupon bond, assuming unchanged (5 per cent) yield to maturity over the 2 years.



Price quotation in this case is much simpler, since the “clean” price and its change also express the change in yields. Example 19 shows that a constant yield means a constant “clean” price, which also clearly expresses the steady nature or stability of the fundamentals. By contrast, the “dirty” price that also contains accrued interest changes continuously, but this change only reflects the mechanical employment of the present value formula and its predictable development.

6 Interest rate risk

After understanding net and gross price, we need only concentrate on the factors that cannot be calculated in advance, i.e. where we face uncertainty. In terms of the factors influencing the total value of the bond, we will only discuss the interest rate and its impact.

Interest rate risk means that the price of the bond changes due to shifts in the interest rate and that – since in the case of bonds interest rate risk is a general phenomenon – the value of our investment changes. This uncertainty is called risk. In this sense, risk is a symmetrical feature, since we take a neutral stance as to whether the bond's value increases or decreases.¹² We will examine how much the value changes, given specific assumptions.

Various risk factors influence the prices of financial instruments, and thus also the prices of bonds. In the case of bonds, however, the most prominent risk factor is interest rate risk. In overly simplified terms, this is the only “component” of the present value formula which moves unpredictably.¹³

In quantifying risks, we always observe how much the price of the asset changes due to a unit change in a particular risk element, i.e. how sensitive the price of the asset in question is to the given risk factor. In the case of interest rate risk, this means the amount by which the bond's price changes due to a unit change in the interest rate. In essence, we attempt to capture the non-linear relationship between yield and price, as illustrated on Chart 10.

¹² The concept of risk and the definitions of the different financial risks are not discussed in this handbook, as we concentrate on the mathematical description of changes in bond prices.

¹³ Of course, this statement is grossly simplified, since in this pure form it is only true in case of a very narrow group of bonds. In fact, even within that group, it only holds true for an even narrower group of government securities, and only for securities with the best credit rating, issued by the most developed countries (e.g. US or German government securities). In addition, in the case of corporate bonds, the key component is credit risk, while in the case of US mortgage-backed securities it is the so-called prepayment risk. But even in such cases, interest rate risk emerges as a risk factor influencing the price of bonds.

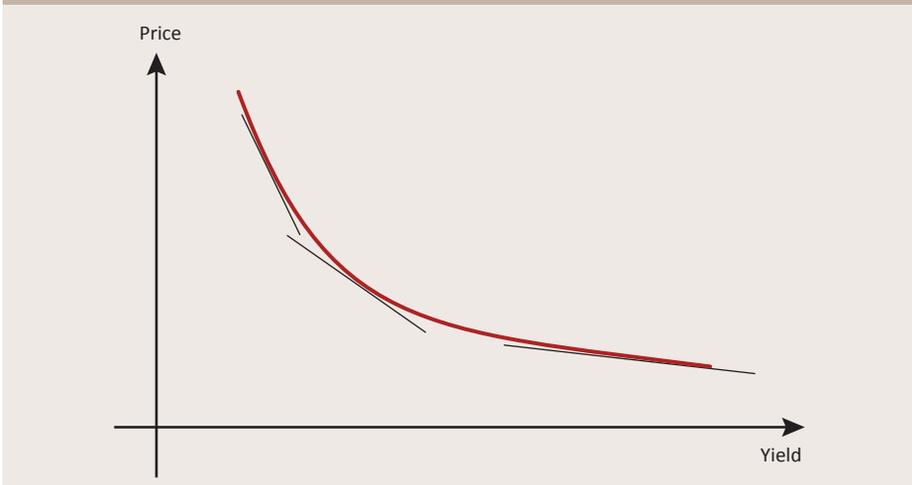
6.1 Duration

The “measure” of bonds’ interest rate risk or interest rate sensitivity is known as duration.

In the past, the most widespread measure was maturity. The longer the bond’s maturity, the higher its interest rate sensitivity. This is true if all other parameters (coupon, yield) are constant, but this measure only takes into account a single cash flow element (the last cash flow), while disregarding the time value of money. For example, is an 11-year, 8-per cent coupon bond more risky than a 10-year, 1-per cent coupon bond? Duration is much more suitable for comparing apples to apples.

There are three interpretations of duration in the literature. According to the first, duration is the slope of the yield-price curve measured at a certain point, i.e. the steepness of the curve’s tangent line. This is perhaps the best way to illustrate the inverse relationship between yield and price. As the yield rises, the price decreases at an ever slower pace as the curve flattens out, i.e. as its slope declines.

Chart 13
Slope of the yield-price curve at different points



The second interpretation is the present value weighted average maturity, which expresses the interest rate risk in years. This interpretation takes into account the fact that a bond has a cash flow even before maturity, and it also captures the time value of money.

$$DUR = \sum_{n=1}^N \frac{PV(CF_n) \cdot n}{P} = 1 \cdot \frac{PV(CF_1)}{P} + 2 \cdot \frac{PV(CF_2)}{P} + \dots + N \cdot \frac{PV(CF_N)}{P}$$

The formula¹⁴ basically compares the present value of the individual cash flow elements to the total present value (i.e. the price of the bond), weighted by the maturity of each cash flow. Thus, we try to determine how soon “we will get back our money”.¹⁵

Example 20: Duration calculation

Calculate the duration of a 5-year, 5-per cent annual coupon bond with an 8 per cent yield to maturity.

<i>n</i>	<i>CF_n</i>	<i>PV(CF_n)</i>	<i>PV(CF_n)/P</i>	<i>n*(PV(CF_n))</i>
1	5	4.63	5.3%	0.05
2	5	4.29	4.9%	0.10
3	5	3.97	4.5%	0.14
4	5	3.68	4.2%	0.17
5	105	71.46	81.2%	4.06
	Total (P)	88.02	Total (DUR)	4.51

The duration of the bond is 4.51 years.

¹⁴ Mathematically, the change is represented in a so-called differential equation. Differentiation, i.e. the description of the magnitude of change, can be used in relationships (functions) that can be approximated well with a linear function, i.e. the graph of which is not different from a straight line within an arbitrarily chosen margin of error. The latter is the tangent of the curve, which was used in the first approach. We use it for describing processes that instead of changing in discrete steps, change constantly from one moment to the next (e.g. they are not broken). In solving engineering problems and in economics, various processes are captured in a similar fashion.

¹⁵ This is because it is important to know whether from a present value of HUF 100, HUF 90 is received in a year and HUF 10 later, or the other way round.

It follows from the above that the higher the coupon, i.e. the larger the cash flow element received before maturity, the greater its weight. In the case of Example 20, the first HUF 5 coupon is represented with a weight of 5.3 per cent, while the second with only 4.9 per cent. Even under the above conditions, the last coupon received at the end of the 5th year and the principal constitute the largest portion of the present value (81.2 per cent), but not as much as if the time value of money was not taken into account and only nominal values were used.

The higher the coupon, the shorter the duration. Of course, the reverse is also true, the smaller the coupon, the longer the duration. In the case of a zero-percent coupon, i.e. zero-coupon bond, duration equals the term to maturity, as the whole present value is comprised of a single cash flow.

Furthermore, it can be stated that the higher the interest rate used for discounting, the larger the role of the cash flows received earlier, as a rising interest rate reduces the present value of later cash flows. In the case of the bond used in Example 20, if the yield is 30 per cent, the weight of the first HUF 5 coupon increases to 9.8 per cent (as the weight of the last cash flow declines to 72.3 per cent), and duration drops to 4.22 years. Of course, this tallies with the fact that the slope of the yield–price curve constantly decreases, i.e. as the yield rises, the pace of the decline in the price decelerates.

Table 2	
Change in duration depending on the individual parameters of the bond	
Maturity ↑	Duration ↑
Coupon ↑	Duration ↓
Yield ↑	Duration ↓

The above interpretation of duration is also called Macaulay-duration. It basically expresses the “average” time during which the bond pays its cash flows, where the weighting of the average is calculated from the present values of the cash flows.

The third approach tries to capture the change in the bond’s price triggered by a certain change in yields. This is expressed by the so-called modified duration.

$$DUR_{\text{mod}} = \frac{DUR}{(1+r)}$$

The question that can be posed here is how much does the price of the bond change in percentage terms if its yield changes by 1 percentage point. This expresses the concept of interest rate sensitivity or interest rate risk in the most clear-cut form.

$$\frac{\Delta P}{P} = (-) \cdot DUR_{\text{mod}} \cdot \Delta r$$

Example 21: Calculation of bond price changes

How much does a bond's price change with a 0.50-per cent rise in the yield, if the modified duration of the bond is 4.28 years and its price is HUF 95?

$$\begin{aligned} \frac{\Delta P}{P} &= (-) \cdot 4.28 \cdot 0.005 = -0.0214 = -2.14\% \\ \Delta P &= 95 \cdot (-0.0214) = -2.033\text{Ft} \end{aligned}$$

The price of the bond decreases by HUF 2.033, i.e. the new price will be HUF 92.967.

6.2 DV01

Bond yields rarely change by 1 percentage point, especially within a day. Therefore, in practice the change is often expressed in the smallest unit of quotation, the so-called *basis point*. One basis point is one-hundredth of a per cent (0.01%).

The price change for the 1-basis point yield change is called the price value of a basis point (PVBP), or the dollar value of a basis point (DV01). Both expressions clearly show what we seek to determine: the change in the value of our investment (be it a bond or a whole bond portfolio) if the yield changes by one basis point.

The main area for the practical application of DV01 is the establishment of so-called duration-neutral positions. In all cases, the aim is to keep the total value of the combined position comprising mostly two instruments unchanged when the interest rate changes. It is neutral because the change in the interest

rate exerts the same impact on the value of both instruments, but since one of them is bought while the other is sold, the value of the combined position remains the same. This is the basic principle in hedging transactions: combining instruments to be hedged and those used for hedging in a way to make the value of the two-element portfolio constant.¹⁶

The significance of the measure is attested by the fact that bond traders, interbank market makers and bond portfolio managers often use this measure to express the amount of risk they can take. It serves as a sort of risk equivalent, as this is the basis for comparison. The interest rate sensitivity of a 2-year interest rate instrument¹⁷ is lower than that of a 10-year instrument, and therefore, in nominal terms, a larger position can be established in the former to run the same amount of interest rate risk.

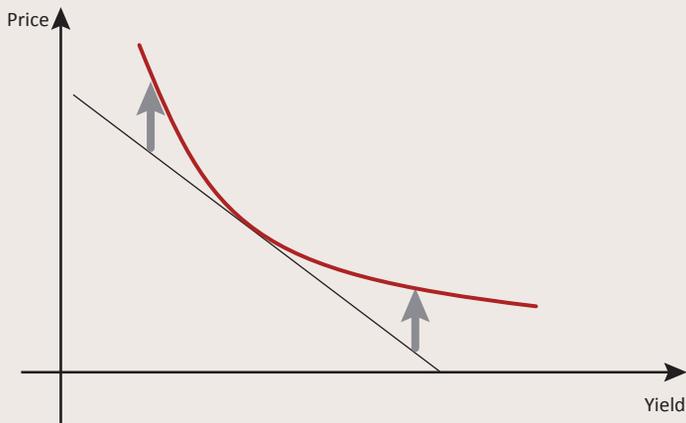
6.3 Convexity

The drawback of estimating the price change using duration is that this method attempts to approximate a fundamentally non-linear relationship with a linear one (see Example 21). Starting out from a certain point, the larger the yield change used as a basis, the greater the difference between the value estimated with the linear equation (tangent) and the actual price calculated with the non-linear formula (curve). Assuming a small change, the estimation is accurate, however, in the case of a greater shift, the “error” steadily increases, since as the yield grows, the pace of the price decline diminishes, and as the yield drops, the pace of the price increase continually increases.

¹⁶ Of course, this is only true in a given moment, since duration and thus also DV01 changes when the yield shifts and even as time passes. It is rare for two bonds to have the same interest rate sensitivity for a long time, and thus hedging ratios must be dynamically modified over time.

¹⁷ These instruments can be either government securities, an interest rate swap or government bond futures.

Chart 14
Estimating bond price changes with duration



The increasing inaccuracy described above is adjusted by the so-called convexity, which shows the change in interest rate sensitivity as the function of the change in yields.¹⁸ When the yield fluctuates, changes can be observed in not only the price but also in the pace of change. Another unit change of the yield induces a price change of different size (a decrease in the case of a yield rise, and vice versa). Convexity is basically used for the more accurate approximation of the curve, and it adjusts for the inaccuracy of the duration measure.

$$\frac{\Delta P}{P} = (-) \cdot DUR_{\text{mod}} \cdot \Delta r + \frac{CONV}{2} \cdot (\Delta r)^2$$

It is important to understand that the “adjustment” part of the above equation always takes a positive value. This is because the slope of the yield-price curve steadily decreases as the yield rises, and therefore the reduction estimated with the duration must be adjusted upwards. Similarly, the slope of the price curve increases as the yield drops, and thus an upwards adjustment is necessary here as well.¹⁹

¹⁸ In mathematical terms, this is the second derivative of the yield-price curve. The first derivative is the change (the steepness of the curve’s tangent), while the second derivative is the change of the change (the change in the steepness of the tangent).

¹⁹ This is true for bonds without an optionality. In the case of a callable bond, the price curve has a concave segment where convexity takes a negative value (see Section 6.4).

The extent of convexity is positively correlated with the dispersion of the cash flow stream. *Ceteris paribus*, the more the cash flows of a bond are dispersed, the greater its convexity. The practical significance of this is that the price of a bond with greater convexity falls less when yields rise. Thus, convexity represents value to bond investors. The extent of the value depends on the current level of yields and the expected market volatility. When a market participant expects greater volatility than currently observed, i.e. larger fluctuations in prices and yields, the value of convexity increases.

6.4 Negative convexity

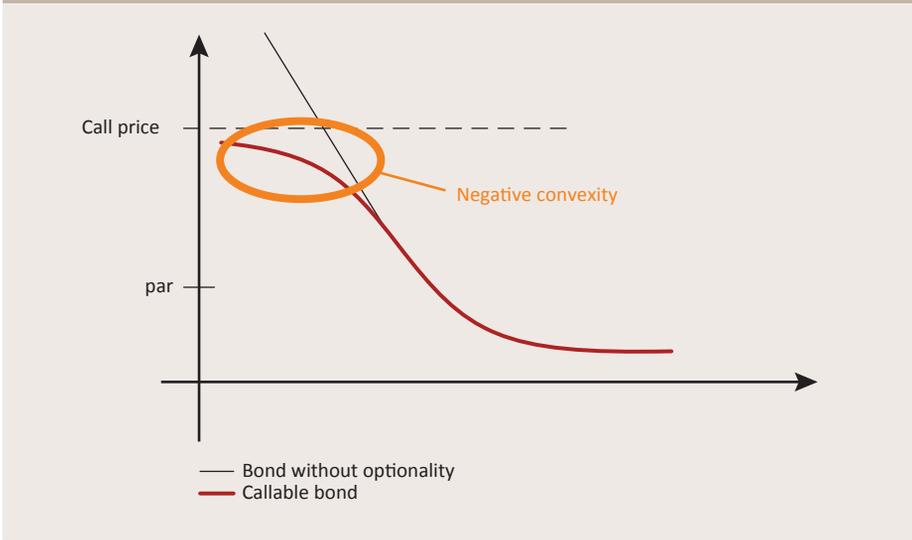
In introducing the concept of convexity, we based our discussion solely on the pricing formula of fixed-rate bonds without an optionality. However, in the case of so-called callable bonds, the issuer has the right to buy back the bond at a predetermined time and at a predetermined price.²⁰ This additional right of the issuer has an impact on the bond's price as well as its dynamics.

The issuer can usually redeem the bond at a price higher than the issue price (call price). When the bond is issued, the coupon rate is typically close to market yields, i.e. the issue price is close to par. In the case of a fixed-coupon bond, the issuer fixes the financing yield until the maturity of the bond. A potential drop in yields, however, provides a good opportunity for issuing bonds at lower yields, and thereby achieving a lower financing rate and lower capital costs. If a company issues callable bonds, it has the option to buy back the bonds, and issue new bonds with lower yields when yields drop and bond prices rise.²¹

²⁰ Countless combinations are possible in this respect. The redemption right may be for a single date, several discrete dates (e.g. for the time of coupon payment in each year), but it may also be a permanent purchase right that can be exercised at any time.

²¹ Of course, this right also comes at a cost, which is included in the yield of the bond. However, the discussion about the option premium is beyond the scope of this handbook.

Chart 15
Price curves of a callable bond and a bond without optionality



Since the issuer can redeem the bond at a predetermined price, the bond's price will not rise even if yields drop further. Therefore, the yield-price curve of callable bonds deviates from the yield-price curve of option-free bonds at a certain point, and instead of continuing to rise at an accelerating pace, it bends back and flattens out towards the call price. The fact that it bends back means that in the case of a further contraction in yields, the price – instead of rising at an ever quicker pace – grows at an ever slower pace. This is negative convexity.²²

²² Negative convexity can be observed in the case of US mortgage-backed securities. Here the borrower of mortgage loans that are packaged into mortgage bonds has the redemption right. In practice, the US market convention is that home buyers may make early repayments of their housing loans at any time and in any amount.

7 Yield curve

The yield curve shows the annualised yield of bonds with different maturities as a function of maturity. The yield curve is also called the term structure of interest rates. The concept itself is very simple, since we merely chart the current yields of papers with different maturities in the maturity-yield space. But there are several other question that need to be clarified before we can make any claims about the yield curve or use it in practice (e.g. for the purpose of discounting).

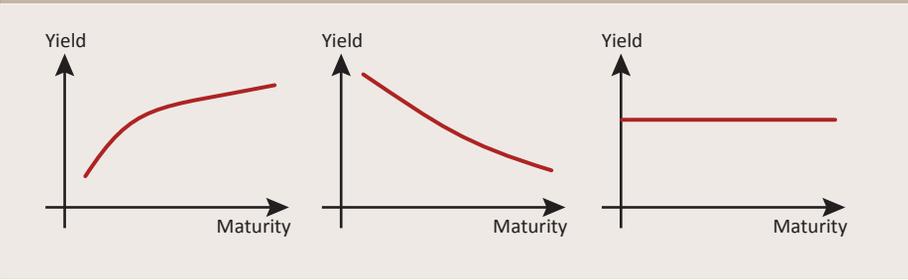
When discussing the yield curve, we always mean a class of bonds with similar credit risk. Thus, we can distinguish the government securities yield curve right away, which has special significance. The reason for this is that in every country the credit risk-free investments are embodied by government securities.²³ By contrast, a corporate bond carries additional (credit) risks, which must be reflected in the yield payable by the company, and this gives us another yield curve.²⁴

The “normal” shape of the yield curve is a curve with a positive slope. This is because if we lend out our money for a longer period (“deposit” it for longer) then we usually expect higher interest for that money. In reality, however, or at least in certain periods, a wide range of forms can be observed. The yield curve can also take an inverted shape, when the yields at the front end of the curve are higher than at the long end, i.e. the slope of the yield curve is negative.

²³ The state has rights (e.g. taxation, printing money) that allow us to assume that it can fulfil its primarily nominal obligations in its own currency at any time, for example the interest and principal payments of government securities.

²⁴ The corporate bond market is not homogenous either, as there are yield curves for individual companies (issuers) that are only characteristic of the given company.

Chart 16
Normal, inverted and flat yield curves



7.1 Spot yield curve

From a practical perspective, the government securities yield curve acts as a point of reference for the pricing of other assets (e.g. bonds, loans). This is the basis when we wish to determine the present value of an arbitrary cash flow. From this perspective, it is irrelevant whether we examine the coupon payment of a bond or the cash flow generated by a project; we simply need an interest rate we can use for discounting the cash flow of that receivable.

Nevertheless, until now we have used a single interest rate in the present value calculation formula, i.e. we did not distinguish between the funds received at different times by their maturity. It was irrelevant whether the given cash flow was received in a year or ten years later, as if it was irrelevant whether we invest for one or ten years. As this is not irrelevant at all, we need a yield curve from which we can derive the different interest rates used for calculating the present value of cash flows received at different times.

However, the government securities yield curve is not suitable for this. First, it is possible that the yields of two government securities with similar maturities are different, for example because they have different coupon rates,²⁵ and second, the bonds' observed (quoted) yield is a yield to maturity, embodying

²⁵ Which causes the convexity of bonds to differ.

the assumption that the cash flow received until the end of the maturity period is reinvested at the yield to maturity.

This is clearly too strong of an assumption if we wish to use this yield curve for the valuation of cash flows received at different times. We need an interest rate that does not incorporate this assumption. And this is the yield to maturity of a zero-coupon bond, also known as the zero-coupon yield or spot rate (S_n).

The coupon rate of a zero-coupon bond is zero, there is no cash flow before maturity, there is nothing to reinvest, therefore no assumptions have to be made regarding the reinvestment rate. The yield curve constructed from zero-coupon yields is called the zero-coupon yield curve or spot curve.

Debt management agencies all over the world issue government securities with various maturities.²⁶ Short-term papers are typically zero-coupon bonds (discount Treasury bills),²⁷ while securities with maturities of over a year are coupon bonds. The yield of zero-coupon securities is the spot rate itself, but the yield to maturity of coupon securities have to be “converted” in order to arrive at the spot rate for the individual maturities and thus a spot curve.

All bonds and all cash flows of a bond (coupons and principal) can be regarded as combination of individual zero-coupon bonds. Progressing gradually along the yield curve, the spot rate for the individual maturities can be derived from the spot rates observed at the front-end of the yield curve and from the observed yields of the coupon bonds. And if the spot rates are known,

²⁶ At the long end of the curve, the 30-year maturity plays an important role (“long bond”), but for example the French and British state regularly market 50-year papers, and there are examples of 100-year issues as well.

²⁷ Any paper that does not have more than one cash flow until maturity can be considered to be a zero-coupon bond. It does not matter if it only pays the face value or the face value plus the last coupon at maturity.

the present value of any cash flow can be calculated by employing the yields corresponding to the relevant maturity. This method is called bootstrapping.²⁸

$$P_0 = \frac{C}{(1+S_1)} + \frac{C}{(1+S_2)^2} + \frac{C}{(1+S_3)^3} + \dots + \frac{C}{(1+S_N)^N} + \frac{M}{(1+S_N)^N}$$

Example 22: Determining the spot curve

Let us assume that the following bonds with annual coupon payments can be observed on the market. Determine the spot curve.

Tenor	Coupon rate	Yield (ytm)	Price
1 év	0.00%	2.00%	98.04
2 év	3.00%	2.50%	100.96
3 év	2.00%	3.00%	97.17

The first bond is a zero-coupon bond maturing in 1 year. Its yield is a spot rate, which is basically a 1-year spot rate:

$$S_1 = 2.00\%$$

Based on the known price of the second bond:

$$100.96 = \frac{3}{(1+0.02)} + \frac{103}{(1+S_2)^2}$$

$$S_2 = 2.51\%$$

Based on the known price of the third bond:

$$97.17 = \frac{2}{(1+0.02)} + \frac{2}{(1+0.0251)^2} + \frac{102}{(1+S_3)^3}$$

$$S_3 = 3.014\%$$

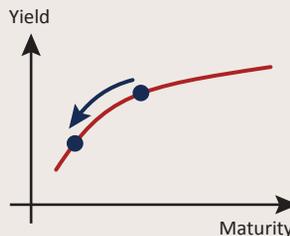
²⁸ In order for the method to work in practice, the cash flows of the bonds have to be traded individually, i.e. the bonds' cash flows need to be "divisible" and "packagable". This way it can be ensured that the sum of the present values of the individual cash flows treated as zero-coupon bonds equals the market price of the "whole" bond. An example for this is the government securities market of the United States, where trading in individual cash flow elements is referred to as STRIPS (separate trading of registered interest and principal of securities).

7.2 “Roll-down”

After determining the spot curve, we can discount all cash flows with the appropriate interest rate without reinvestment assumptions. Nonetheless, it is also important to understand what happens to the bond price with the passage of time assuming an unchanged yield curve. In the case of the pull-to-par phenomenon, we examined the same dynamics, but there we made an additional assumption, namely that the yield curve was flat. That is, it is not just that the yield curve remains unchanged, but that the same yield is used during the present value calculation in the next period, too. Now we go a step further, and examine the bond price behaviour in the context of an arbitrary yield curve.

For example, in the case of a “normal” yield curve with a positive slope, the price of the bond will not only change due to the pull-to-par phenomenon, but also because with the passage of time, a different yield is used for the present value calculation. In the case of a curve with a positive slope, as the remaining maturity of a bond decreases, the interest rate used for discounting drops, which exerts an additional impact on the price of the bond. This is called the roll-down effect, since the bond “trickles” or rolls down the yield curve.

Chart 17
Roll-down along the yield curve



Example 23: The roll-down effect

How does the price of the 2-year zero-coupon bond change over a year, if the 1-year and 2-year spot rates are 4.00 and 5.00 per cent, respectively, and this yield remains the same during the period? How much will the holding period return of the bond be over this 1 year? How much will it yield during the second year?

$$P_0 = \frac{100}{(1+0.05)^2} = 90.7029$$

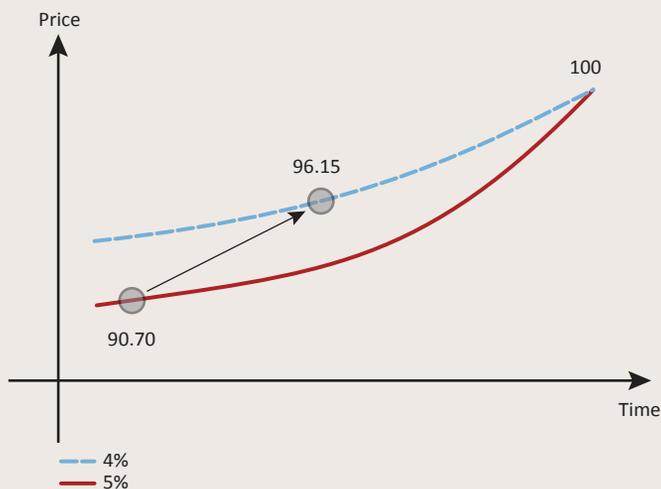
$$P_1 = \frac{100}{(1+0.04)} = 96.1538$$

$$HPR_1 = \frac{96.1538 - 90.7029}{90.7029} = 0.0601 \text{ i.e. } 6.01\%$$

$$HPR_2 = \frac{100 - 96.1538}{96.1538} = 0.0400 \text{ i.e. } 4.00\%$$

The lower discount rate means a higher bond price and thus a higher holding period return. If the paper in Example 23 is held for two years (until maturity), we can realise an annual yield of 5 per cent. However, due to the curvature

Chart 18
Pull-to-par curves for the 4-per cent and 5-per cent yields and the roll-down effect



of the yield curve, this yield is not distributed evenly. The total yield will be 6.01 per cent in the first year and 4 per cent in the second.

Of course, the roll-down effect is exerted on all points of the yield curve, irrespective of its shape. With an inverted yield curve, the effect is the opposite, since in that case the yield rises as the maturity shortens, which results in a lower price and thus a lower holding period return (“roll-up”).

7.3 Forward yields

In order to better understand the yield curve and bond investments, the so-called forward yield must be defined. The forward yield is essentially a future yield derived from the spot curve expressing yield expectations for a period beginning in the future.

First, let us assume that we can choose between two possible investments. In Scenario A) we can invest for two full years at the 2-year spot rate, while in Scenario B) we can first invest for one year at the 1-year spot rate, then in the second year we can invest for another year, but at a currently unknown yield level.

How can we choose between the two investment options? The answer depends on our expectations about the spot rate one year from now, and thus the holding period return for the 2 years in total. It is therefore a subjective matter. However, we could also ask what 1-year spot rate would make us indifferent about choosing between the two options. Obviously, we will be indifferent if the two investment opportunities provide the same holding period return over the 2 full years. The yield ensuring that the two investments are equal from a return perspective is called the forward yield.

$$\begin{aligned}HPR(A) &= HPR(B) \\(1+S_2)^2 &= (1+S_1) \cdot (1+F)\end{aligned}$$

Example 24: Forward yield calculation

Let us assume that the 1-year and 2-year spot rates are 4.00 and 5.00 per cent, respectively. How much is the 1-year forward yield 1 year from now?

$$(1+0.05)^2 = (1+0.04) \cdot (1+F)$$

$$F = \frac{(1+0.05)^2}{(1+0.04)} - 1$$

$$F = 0.0601, \text{ i.e. } 6.01\%$$

We can rephrase this by saying that we will be indifferent with respect to the two investment opportunities if it can be guaranteed that in 1 year the bond will have this 6.01 per cent spot rate. This forward yield is the 1-year forward 1-year rate *implied* by the spot curve.²⁹

Note that in Example 23, the holding period return in the first period is the same as the 1-year forward 1-year yield calculated in Example 24. Actually, this forward yield (6.01 per cent) is earned in the first year, this is what the investment yields over the period. That is why it is called implied forward yield, because it is implied for the given period from the current (spot) curve. This holds true for any period along the yield curve. If the yield curve's level and shape remain unchanged, the forward yield will be earned.

Similarly, the implied forward yield over an arbitrary timeframe can be derived from the spot curve for any arbitrary period. Therefore, the following general formula can be used to calculate the 1-year forward yield 1, 2... 10 years from now, the 2-year forward yield 1, 2... 10 years from now, as well as the 2-, 3-,

²⁹ Its market notation may also be 1yr1yr, 1y1y or even 1*1.

10-year forward yield 5 years from now by computing the annualised $(n-m)$ -year forward yield in m years $(F_{m,n})$.³⁰

$$(1+F_{m,n})^{n-m} \cdot (1+S_m)^m = (1+S_n)^n$$

$$F_{m,n} = {}^{n-m}\sqrt{\frac{(1+S_n)^n}{(1+S_m)^m}} - 1$$

Example 25: Calculation of forward yields for different periods

Let us assume that the 1-year, 2-year and 3-year spot rates are 4.00, 5.50 and 5.50 per cent, respectively. How much is the 1-year forward yield 2 years from now $(F_{2,3})$? How much is the 2-year forward yield 1 year from now $(F_{1,3})$?

$$(1+0.055)^3 = (1+0.05)^2 (1+F_{2,3})$$

$$F_{2,3} = \frac{(1+0.055)^3}{(1+0.05)^2} - 1$$

$$F_{2,3} = 0.0651, \text{ i.e. } 6.51\%$$

$$(1+0.055)^3 = (1+0.04) \cdot (1+F_{1,3})$$

$$F_{1,3} = \sqrt{\frac{(1+0.055)^3}{(1+0.04)}} - 1$$

$$F_{1,3} = 0.0626, \text{ i.e. } 6.26\%$$

³⁰ There are various notations in the literature. For example ${}_mF_n$ could denote the m -year forward yield in n years as well. The logic behind the notation used here lies in the two spot rates (m and n), between which we would like to calculate the forward yield. Therefore, this refers to two points on the yield curve, and the length of the forward period can be derived from this $(n-m)$.

7.4 Yield curve theories

Several theories have been developed to explain the yield curve's shape and its changes. There are three basic factors influencing the yield curve, and the dynamics between the three – i.e. which of them exerts the strongest effect – change over time and along the yield curve.

The first factor is the development of market expectations. This is basically a fundamental approach, since the price of money (interest) is defined partly by the issuer of the money, i.e. the central bank itself. This means that the expectations about monetary policy are key in shaping the yield curve in every economy.

One of the expectation theories is the so-called pure expectations theory. According to this, the forward yield reflects nothing but the expected future spot rate. And the total yield curve reflects nothing but the market expectations about future short-term yields. In this framework, a rising yield curve means that the market expects short-term yields to rise, which is solely derived from the formula used for calculating the forward yield. A flat yield curve indicates stable short-term yields, while an inverted yield curve points to a decline in yields. In addition, market expectations do not only concern yields but also the slope of the yield curve. The expectations about the slope are reflected in the curvature of the current curve.

One of the consequences of the expectations hypothesis is that, irrespective of the length of the investment horizon, we can be sure about the return earned for a given period, regardless of the maturity of the bond. The higher yield of longer-term bonds is not realised by the investors, because realised yields are offset by the increasing yields indicated by the upward sloping yield curve.

Naturally, experience suggests otherwise. Expectations theories are complemented by risk premium theories in the explanation of the yield curve. These theories are based on the fact that bond prices not only reflect expectations but also risks, and therefore they need to offer a premium over the risk-free yield to make investors willing to invest.

Various theories have been proposed, each reflecting different views about the direction of the premium's effect (positive or negative), the factors determining the premium and the potentially changing nature of the relationship between the explanatory factors and the premium.

The liquidity premium theory is based on the tenet that due to uncertainty about the future, investors are more inclined to invest their capital for shorter periods, i.e. they are only willing to buy longer-term bonds with higher yields or with a liquidity premium. This theory explains the fact that the yield curve is usually rising by claiming that yields for longer maturities contain a liquidity premium. In this framework, the rising yield curve does not necessarily mean that short-term yields will move on an upward path, since the positive slope only reflects the higher risk of longer-term securities.

The preferred habitat theory must also be discussed here. It argues that the risk premium may even decrease as maturity increases, since for investors with long obligations (e.g. pension funds) longer-term securities pose less risk, and therefore these investors would expect a liquidity premium precisely when holding shorter-term securities. They are only willing to invest on the short end of the yield curve if they are compensated for the risk they take in the form of a higher premium.

Other effects can also be identified. An example for this is the convexity bias. Positive convexity is a desirable feature, since it makes bond prices go down at an ever decreasing pace in the context of rising yields, while in an environment of decreasing yields, it makes bond prices increase at an ever greater pace; therefore, it has a positive effect on the realised return. Depending on market expectations, namely the volatility anticipated by market participants, the value of convexity may increase, in which case more convex bonds are traded at a higher price, i.e. with lower yields.

In practice, the above-mentioned factors do not shape the yield curve independently but all simultaneously, and thus it is rare that a single theory can explain the current shape of the yield curve. Actually, the above theories "live" together. On the one hand, they can only interpret changes together, and on the other hand different theories may apply to different segments of

the curve. For example, the short end of the yield curve is basically shaped by changes in short-term interest rates, i.e. expectations about monetary policy, and the different risk premiums have much less influence there than for instance on the long end of the yield curve, where we can more frequently observe that the supply and demand conditions characteristic of a certain period dominate market movements.

8 Trading along the yield curve

The movement of the yield curve influences bond prices, and thus also the return realised by investors. Basically three parameters change, and in practice they can shift discretely or at the same time in any combination. The level, slope and curvature of the yield curve can all change.

Depending on their expectations, investors can profit from the change in all three if their expectations are realised. We will present the possible positions below using three US government securities: 2-year, 5-year and 10-year on-the-run government bonds.³¹

Name	Term to maturity (yrs)	Coupon	Yield	Duration	DV01
T 0% 11/17	2	0.875%	0.94%	1.978	\$198
T 1% 11/20	5	1.625%	1.67%	4.782	\$477
T 2% 11/25	10	2.250%	2.23%	8.879	\$890

** The DV01 is for a face value of USD 1 million*

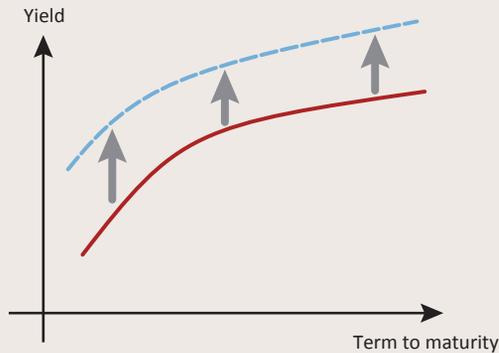
8.1 Change in the level of the yield curve

A change in the level of the yield curve means a parallel shift, either upwards or downwards. If the curve exhibits a parallel shift upwards, i.e. a general rise in yields is observed, then bond prices fall.³² If the curve shifts downwards, i.e. yields drop, bond prices increase.

³¹ November 2015.

³² Without knowing the exact magnitude of change, it is difficult to say whether this negative effect offsets, for example, the daily coupon yield of the bond.

Chart 19
Parallel shift in the yield curve



Accordingly, in the case of a decline in yields we take a long position, i.e. we buy bonds, while in the case of a rise in yields we take a short position, i.e. we sell bonds.³³ With a long position, we profit from the rise in prices and incur losses when they decline, while with a short position the opposite is true, i.e. we profit from a fall in prices and incur losses when they increase.

In view of the fact that the yield curve changes to the same extent over its entire length, it is irrelevant whether we take the position at the front or long end. What matters is that the interest rate sensitivity (duration) of longer-term bonds is greater; therefore, if we buy bonds with the same face value, we take on more risk in the case of longer-term bonds. And conversely, if we wish to assume the same interest rate risk, we should buy less from longer-term securities.

$$\text{Notional position size} = \frac{\text{Targeted amount of risk (DV01)}}{\text{DV01 for a face value USD1 million}}$$

³³ In this handbook, we disregard whether this means a so-called *outright* purchase or selling or taking a position against a benchmark. The goal here is to understand the mathematical relationships.

Example 26: Taking positions in expectation of a parallel shift in the yield curve

We expect a downward parallel shift in the yield curve. The maximum interest rate risk we can take is DV01=100,000 dollars. What is the maximum nominal amount that we can buy from the government securities of different maturities (see Table 3)?

Name	DV01	Maximum notional position (mio USD)
T 0% 11/17	\$198	505
T 1% 11/20	\$477	210
T 2% 11/25	\$890	112

In fact, as the only thing that matters is the amount of interest rate risk we take, we can take the risk at any point on the yield curve and in any combination. When the position is concentrated around one point, it is a so-called bullet position. We can for example directly buy the 5-year paper. The same interest rate risk can be “mixed” by combining a short-term and a long-term bond. This is the so-called barbell strategy. Purchasing 2-year and 10-year bonds with the appropriate weights entails exactly the same interest rate risk.

Example 27: Bullet and barbell positions

What should be the face value of the 5-year bond we buy if we wish to take a long position of USD 25,000 DV01? How much should we buy from the 2-year and 10-year securities if we would like to achieve this in a barbell structure with a 50-50 per cent division (i.e. equal face value)? (Let us use the figures from Example 26 for the calculation.)

$$\frac{25000}{477} \approx \text{USD } 52.5 \text{ million}$$

$$\frac{25000}{198+890} \approx \text{USD } 23 \text{ million}$$

This means that we either buy the 5-year bond in the amount of USD 52.5 million, or each of the 2-year and 10-year bonds with face value of USD 23 million each.

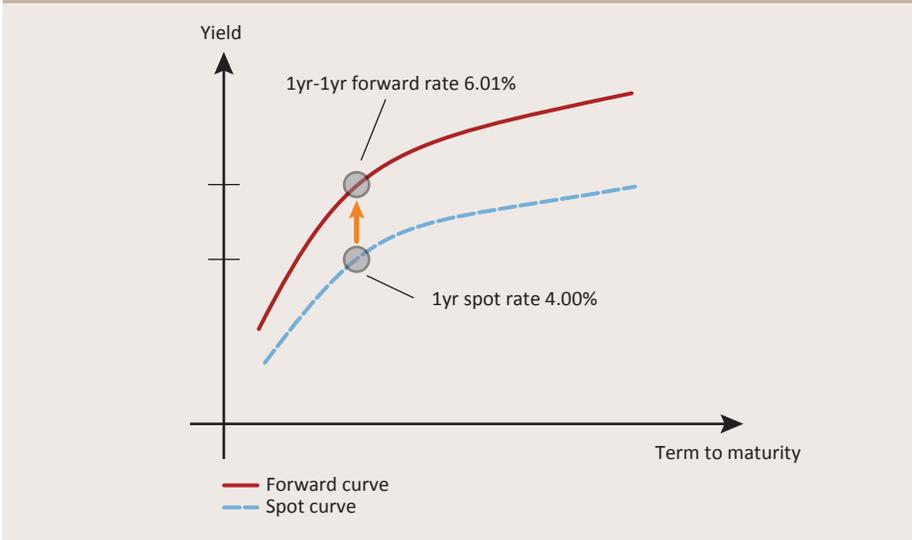
When we take on interest rate risk and decide whether it should be long or short position, our decision is assisted by the forward yields. Each forward yield can be regarded as a break-even rate. Let us assume that we can choose between two investments: we either buy 1-year or 2-year securities. Based on the figures in Example 24, we calculated how much the return on our investment will be over the next year, provided that market conditions remain unchanged. The return on the 1-year paper will be simply 4 per cent, while we can earn the forward yield on the 2-year paper, i.e. 6.01 per cent. If we expect this, then the decision is quite straightforward: we should buy the 2-year paper, as it is worth “going out” on the curve and taking more duration risk.

How much will the return be if the market expectations implied by the forward rate are realised? In such a case, the return for the first year will be exactly 4 per cent for both papers. This tallies with the claim of the pure expectations hypothesis, namely that irrespective of the maturity of our investment, we will realise the same return over the same investment horizon.

The reverse can also be examined: when is it worth buying the 1-year bond instead of the 2-year bond, i.e. taking less interest rate risk and “shortening” the position (reducing interest rate sensitivity or duration)? If the yield curve remains unchanged over the investment period, we will clearly make losses, because we only realise a 4-per cent return instead of 6.01 per cent. But if yields rise, the return on the 2-year paper will be lower (the return of the 1-year paper will certainly be 4 per cent because we hold that until maturity).

Therefore, the question is how much should the 2-year yield rise so that we realise the same return on the two investments? What is the so-called break-even rate? Obviously, the answer is that the 1-year yield can increase to 6.01 per cent (from the current 4 per cent). This means that in the case of a short position, the 1-year yield must rise more than 200 bps, so that it is worth taking a short position, since this amount of increase is already “priced in”. Such a rise is already part of the market expectations.

Chart 20
Spot and forward yield curve



It is important to understand that what counts is not what we expect in absolute terms (a decline or increase in yields), but what we expect compared to what is already priced in. Using the above example, the market has already priced in an interest rate increase by the central bank of more than 200 basis points for the next year. We have to decide how much more interest rate rise can we expect relative to this. If we expect more, then we should take a short position, since the realised return on the 2-year paper will be lower than that of the 1-year paper due to the rise in yields over what is implied by the forward rate, but if we expect less, we should take a long position, since the yield premium of the 2-year paper will compensate us for the negative impact of otherwise rising yields.

8.2 Change in the slope of the yield curve

The change in the slope of the yield curve means that the yield difference between two points of the curve changes. These can be two arbitrary points, and therefore we must specify which two points we are referring to. When talking about the slope of the yield curve in general, we typically mean the difference between the 2-year and the 10-year yield. Furthermore, as the yield

curve “normally” has a positive slope, the slope is expressed by subtracting the 2-year yield from the 10-year yield. In the case of the government securities used as examples, the 2-10-year slope is 1.30 percentage points, i.e. 130 basis points.³⁴

Should this yield spread rise, the curve becomes steeper, while if it decreases, the curve becomes flatter. Accordingly, the profitable position taken in the case of a rising slope is the so-called steepener position, while in the case of a flattening slope it is the so-called flattener position. From this perspective it is irrelevant which element of the yield spread changes (or whether both of them change), the decisive factor is the relative change.

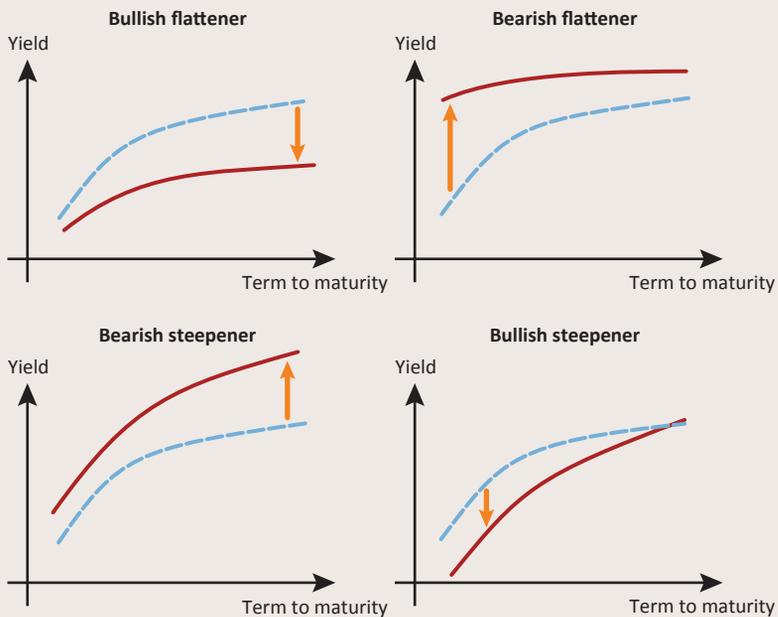
Table 4	
Relative yield changes: cases of the yield curve flattening	
10-year yield	2-year yield
decreases	unchanged
decreases	increases
decreases	decreases less
unchanged	increases
increases	increases more

In the case of a flattener position, we expect the yield of the 10-year bond to fall relatively more than that of the 2-year bond. When yields decline, we hold a long position, whereas when yields rise, we hold a short one, and therefore we sell the 2-year bond and buy the 10-year paper. In the case of the steepener position, the opposite happens in every respect: due to the relative rise in the 10-year yield, we sell the 10-year bond (short) and we buy the 2-year paper (long).

As already noted, different changes in the yield curve may happen at the same time. It is possible that not only the slope of the curve changes, but also its level. If the change in slope happens in an environment of falling yields, the positions are called bullish flattener and bullish steepener, while in the case of a rising yield curve they are referred to as bearish flattener and bearish steepener.

³⁴ Of course, the slope can be calculated between any two points, but there are a few key segments. In addition to the “2-10-year” segment, the 10-30-year segment has also substantial significance for investors focusing on the long end of the yield curve, such as pension funds.

Chart 21
Changes in the slope of the yield curve



However, when we position ourselves only for the change in the slope of the yield curve, the change in its level is irrelevant. Or more precisely, we have to position ourselves in a way to make it irrelevant. To this end, we have to take some form of a neutral position. In this case, we assume a so-called duration-neutral position, because our goal is to make the interest rate sensitivity of the combined position zero.

Since we take a short position on one point and a long one on another, their size has to be calibrated so that the interest rate sensitivity of the long position equals that of the short position. Thus in the case of a parallel shift in the yield curve, the change in the yield will be the same in both points (i.e. their difference remains the same), and since the interest rate sensitivities of the two points are equal, the value of both positions will exhibit the same change. For example, when yields rise, the price of both bonds falls, but on the short

position we profit from the drop, while we make losses on the long position, and the two changes cancel each other out.

Example 28: Positioning for changes in the slope of the yield curve

We wish to take on a 2-10-year duration-neutral flattener position. How much should the face value of the government securities sold and bought at the 2-year and 10-year points of the yield curve be if we can hold up to USD 20,000 DV01 positions on the individual legs?

$$\frac{20000}{198} \approx \text{USD } 101 \text{ million}$$

$$\frac{20000}{890} \approx \text{USD } 22.5 \text{ million}$$

In order to achieve the duration-neutral flattener position, we should sell 2-year papers with a face value of USD 101 million and buy 10-year bonds with a face value of USD 22.5 million.³⁵

As we have pointed out in discussing interest rate sensitivity, duration only gives an appropriate estimate of the magnitude of the actual price change in the case of a small yield movement. In view of the fact that the value of the duration changes with the change in the yield, the interest rate sensitivities of the two positions will change. The duration of the 2- and 10-year papers are likely to exhibit different dynamics, and therefore the position has to be recalibrated from time to time. In addition, with the passage of time, as the remaining maturities of the bonds shorten, their interest rate sensitivity will decrease. Thus, even if the yield curve remains unchanged, the curve position will be less and less duration-neutral.

³⁵ The remaining USD 77.5 million is invested in a so-called cash instrument with a very short maturity and zero interest rate sensitivity (e.g. 1-month deposit).

8.3 Change in the curvature of the yield curve

Two points are not sufficient for determining the curvature of the yield curve, because they only define the slope of the yield curve (or that of its given segment). We need three points for determining the curvature. With this, we basically express the shape of the yield curve between two points. Similar to slope, any combination is possible along the curve, but the 2-5-10-year and the 2-10-30-year combinations are key.

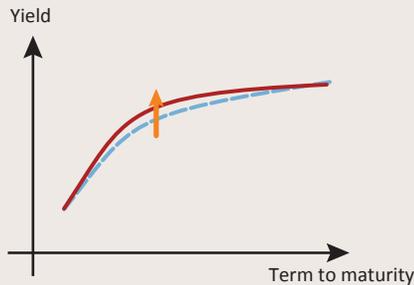
The change in curvature means that the yield curve's shape will be more curved or straighter between the two extreme points. This change can be broken down to the change in the slope of two segments. For example we can regard the 2-5-10-year curvature to be composed of a 2-5-year and a 5-10-year segment, in which case the curvature will be the relative change between the 2-5-year and the 5-10-year slopes. For example as the yield curve becomes smoother, it means that the 2-5-year segment flattens, while the 5-10-year segment steepens.

With respect to positioning, this is the combination of a bullet and a barbell strategy, where the segment's body is the bullet position, and its two wings are the barbell position. That is why this combination is also called a butterfly position. If we expect the curve to become even more curved, then the middle of the curve is sold, since that is where yields will rise (relatively) compared to the wings, which are bought.

A butterfly strategy can be conceived as being the combination of two curve positions. For example a steepener position (2-year buying + 5-year selling) and a flattener position (5-year selling + 10-year buying) are taken. It can be seen that the above-mentioned two methods produce the same result: the middle of the curve is sold, while the wings are bought.

Furthermore, the butterfly strategy can be employed in several constructions, but similar to the slope strategies, the typical approach is duration-neutral. Of course, here it is also true that the strategy's interest rate sensitivity will only be zero in the case of small parallel yield movements.

Chart 22
Change in the curvature of the yield curve



Example 29: Duration-neutral curve position

What should be the proportion of the 2-year, 5-year and 10-year points bought and sold in order to achieve a duration-neutral curve position?

$$1 \cdot 477 = w_{10} \cdot 890 + w_2 \cdot 198, \text{ where } w_{10} \text{ and } w_2 \text{ are the weights of the 10/2yr papers}$$

$$w_{10} + w_2 = 1$$

$$w_{10} \approx 0,40 \text{ and } w_2 \approx 0,60$$

This means that for example if USD 100 million of 5-year bonds are sold, USD 60 million of 2-year papers and USD 40 million of 10-year papers must be bought.

Note that the curve position in Example 29 is not only duration-neutral but also cash-neutral. We can also observe that the different segments have different interest rate sensitivities. The 2-5-year steepener position is also duration-neutral, but the DV01 for each leg for a face value of USD 1 million is only USD $0.6 \cdot 198 \approx 119$, while in the case of the 5-10-year flattener position it is USD $0.4 \cdot 890 \approx 356$. It follows from this that although the strategy contains a curvature element, it is fundamentally a 5-10-year flattener position.

Should we wish to adjust for this distortion in order to approximate the clean curve position, the same duration has to be allocated for the 2-5-year and the 5-10-year slope positions. This means that if we sell 5-year securities with DV01 of USD 477, $USD\ 477/2=238.5$ of interest rate risk has to be bought on the two “wings”. Naturally, in such a scenario the position will not be cash-neutral anymore: if 5-year bonds with a face value of USD 100 million are sold, we have to buy 2-year papers amounting to USD 120 million and 10-year papers amounting to USD 27 million, for which we have to sell 47 million in cash (borrowing).

The clean curve position means that the only thing influencing the profit and loss of the position is the change in the curvature. We can achieve a completely clean curve position if the durations of the two wings are positioned the same distance away from the bullet duration (in this case the duration of the 5-year point). Any asymmetry, steepening or flattening, distorts the position, as in Example 29. In our case, the 2-5-year duration difference is 2.8 years, while the 5-10-year duration difference is 4 years, and therefore the asymmetry is not substantial, but the “wing” and cash positions still need to be adjusted somewhat to achieve a clean curve position: in addition to the USD 100 million of the 5-year short position, USD 61 million in cash has to be sold in order to buy USD 138 million of 2-year bonds and USD 23 million of 10-year papers.

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