Solution Methods for Economies With Large Shocks

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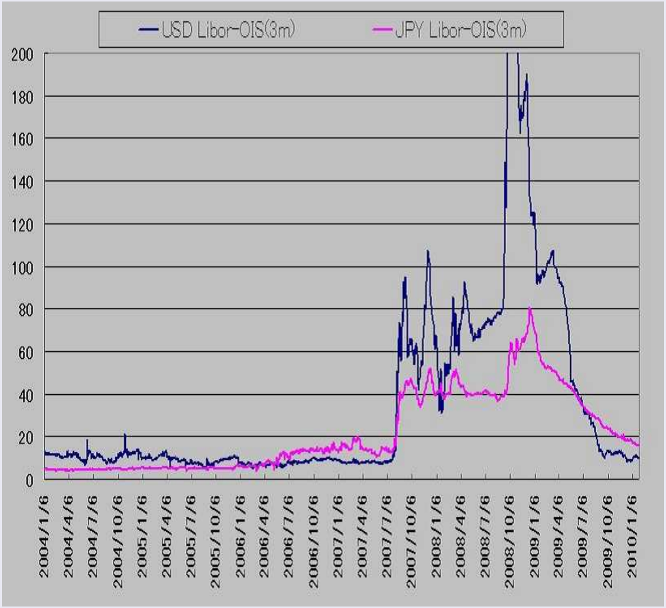
Historical data for USD&JPY Libor-OIS spread



Source of last graph

Masaaki Fujii , Yasufumi Shimada , Akihiko Takahashi:
"On the Term Structure of Interest Rates with Basis Spreads, Collateral and Multiple Currencies"
Conference presentation.

Outline

Solution Methods for DSGE Models

1. Workhorse: linearization around steady state
   Severe limitations, most importantly: Certainty equivalence solution: no effect of uncertainty on behavior

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   Makes sense if local information around steady state is sufficient to recover global solution.

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   Severe limitations, most importantly: Certainty equivalence solution: no effect of uncertainty on behavior
2. Next step: higher order perturbation around steady state (available in Dynare).
   Local method.
   Makes sense if local information around steady state is sufficient to recover global solution.
3. General approach: global methods ("projection methods", "weighted residual methods").
   1. Has been around since Judd (1992).
   2. What's new?
   3. What's important?

Plan

1. Solution techniques: overview.
2. Some examples: big shocks make a difference.
3. Global methods: problems and tricks to solve them.
4. Examples in detail:
    - Simple NK model with zero lower bound.
    - OLG model with portfolio choice.
5. A toolkit to make things easy (easier? too easy?).

Handling Small Shocks: the Concept of Analyticity

Analyticity: local information (value and all derivatives) pins down function globally.
An analytic function is described by its Taylor expansion:

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k \tag{1}$$

Examples:

- exponential function: (1) is valid for all x_0 and x.
  This function has infinite *convergence radius*.
- logarithmic function: convergence radius of (1) is finite.
  Example: $x_0 = 1$. (1) only converges for $|x - 1| < 1$.
  [Still, value of log at any x can be obtained from information at $x_0 = 1$, using Taylor expansions in overlapping circles.]

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2. the "commonly used" functions are all analytic,
3. except for functions like max, absolute value etc (functions with kinks).

Analyticity: consequences for solving models

- If it holds, perturbation methods (local information around deterministic steady state) can be used, although they may not be optimal.
- If not, global methods necessary.

Analyticity breaks down because of

- inequality constraints (occasionally binding)
- regime switching
- etc.

Perturbation Solution

The solution $y(x)$ depends parametrically on the standard deviation of shocks, σ.
Write this as $y(x;\sigma)$.
The perturbation approach approximates this as (scalar case)

$$\begin{aligned} y(x;\sigma) \approx y^* &+ y_x(x^*;0)(x-x^*) + y_\sigma(x^*;0)\sigma \\ &+ \frac{1}{2}y_{xx}(x^*;0)(x-x^*)^2 + \frac{1}{2}y_{\sigma\sigma}(x^*;0)\sigma^2 \\ &+ y_{x\sigma}(x^*;0)(x-x^*)\sigma \\ &+ \dots \end{aligned}$$

Advantages of Local (Perturbation) Methods

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3. Are difficult to implement, but
4. a toolkit is available: Dynare.

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1. high-order approximation may be necessary to get sufficient accuracy;
2. approximate solution may be unstable.
   Proposed solution: "pruning". (Kim, Kim, Schaumburg, and Sims 2008; Lan and Meyer-Gohde 2013; Andreasen, Fernndez-Villaverde, and Rubio-Ramrez 2013)

Use Analytic Approximations?

- Continuous functions can be approximated by analytic functions with arbibrary precision.

Problems:

- Inequality constraints are a simple, intuitive modeling feature.
- In high-dimensional (and even medium-dimensional) applications, only very low-order approximations possible.

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My conclusion: we have to live with kinks, and try to compute solutions with kinks.

Global (Projection) Methods

Example: RBC model

- 2 States:
  1. capital k_t
  2. technology z_t
- Solution: consumption function $C(k_t, z_t)$ and labor supply $L(k_t, z_t)$. They satisfy the Euler equation

$$U_c(C(k_t, z_t), L(k_t, z_t)) = \beta \, \mathrm{E}_t \left[(1 + F_k(k_{t+1}, z_{t+1}) - \delta) U_c(C(k_{t+1}, z_{t+1}), L(k_{t+1}, z_{t+1})) \right] \quad (2)$$

and the labor supply equation

$$F_L(k_t, z_t) = -\frac{U_L(k_t, z_t)}{U_c(k_t, z_t)} \quad (3)$$

Projection Solution, RBC Model

- Approximate

$$C(k,z) \approx \sum_{i=1}^{n} \gamma_i^C \varphi_i(k,x)$$

$$L(k,z) \approx \sum_{i=1}^{n} \gamma_i^L \varphi_i(k,x) \tag{4}$$

where
  - φ_i are known basis functions
  - γ_i^C and γ_i^L are undetermined coefficients.
- Choose γ_i^C and γ_i^L such that Euler equation (2) and labor supply equation (3) are satisfied at a set of grid points

$$(k_i, z_i), \qquad i = 1, \ldots, n \tag{5}$$

(Collocation method.)

Project Method, General Model

Solution: $y(x)$, where x is state vector. Has to satisfy the functional equation

$$\mathrm{E}_t(x_t, y_t, x_{t+1}, y_{t+1}, \epsilon_{t+1}) = 0 \tag{6}$$

Approximate

$$y(x) \approx \sum_{i=1}^{n} \gamma_i \varphi_i(x) \tag{7}$$

Choose γ_i s.t. $\mathcal{F}(x_i, y) = 0$ is satisfied at a set of grid points

$$x_i, \qquad i = 1, \ldots, n \tag{8}$$

Big system of nonlinear equations!

Big Shocks matter in strongly nonlinear models

- Inequality constraints
- More general: strong asymmetries, kinks in functions (capital adjustment cost upwards vs. downwards)
- Asset choice: distribution of shocks essential
- Models of heterogeneous agents, lumpy decision: density at threshold matters

My focus:

1. Handling occasionally binding constraints.
2. Models with portfolio constraints
3. Efficient implementation.

Examples in the literature

1. Petrosky-Nadeau and Zhang (2013a) and Petrosky-Nadeau and Zhang (2013b): Explaining unemployment crises
2. Fernndez-Villaverde et al. (2012): “Adventures at the zero lower bound”
3. Brunnermeier and Sannikov (2012): “A macroeconomic model with a financial sector.”

Solving the Labor Market Matching Model Accurately

Petrosky-Nadeau and Zhang (2013a) consider the DMP model with the calibration of Hagedorn and Manovskii (2008).

This calibration has a small job surplus and leads to large unemployment fluctuations.

Petrosky-Nadeau and Zhang (2013a) find:

- Model has strong nonlinearities:
  - much stronger impulse responses in recessions than in booms.
  - asymmetries in impulse responses

  which are not captured by log-linearized solution.
- Accurately solved, the model fails to explain labor market data.
- Projection method very accurate.
- Second-order perturbation improves on log-linearization.
  But solution is closer to log-lin. than to projection.
  They don't examine higher-order perturbations.

Conclusion: exact solution can matter for the qualitative properties of the model.

Explaining unemployment crises

Petrosky-Nadeau and Zhang (2013b)

- Empirical finding:
  1. unconditional frequency of crisis (UR about 20%) is 2–3%
  2. crisis state very persistent (0.89 monthly)
- Model: labor market matching model (Mortensen and Pissarides 1994; Andolfatto 1996; Merz 1995) with
  1. credible bargaining (Hall and Milgrom 2008): outside option in bargaining is delay, not breaking up the relationship
     $\Longrightarrow$ small feedback from unemployment to wages
  2. vacancy posting costs have a fixed component, additional to cost proportional to number of vacancies (Mortensen and Nagypal 2007; Pissarides 2009)

  can explain unemployment dynamics, including Great Depression. Strong nonlinearities in the model:

  - new matches are product of unemployment and vacancies
  - Inequality constraint: vacancy formation cannot be negative

Solution Method in Petrosky-Nadeau/Zhang

Critical equation:

$$\frac{\kappa_t}{\theta_t} - \lambda(N_t, X_t) =$$
$$\beta \, \mathrm{E}_t \left[X_{t+1} - W_{t+1} + (1-s) \left(\frac{\kappa_{t+1}}{\theta_{t+1}} - \lambda(N_{t+1}, X_{t+1}) \right) \right] \quad (9)$$

- Two states:
  1. employment N
  2. productivity X
- λ is Lagrange mutliplier for inequality constraint $q_t V_t \geq 0$.
- Approximation:
  1. 17 grid points in productivity (discrete, exogenous)
  2. spline approximation with 45 grid points in employment.
- Approximate not $q(N, X)$ and $V(N, X)$, but rhs of (9) (parameterized expectations), to handle kink.

Adventures at the Zero Lower Bound

Fernndez-Villaverde et al. (2012):

- New Keynesian framework:
  - Intermediate good producers with Calvo price setting
  - Monetary policy: Taylor rule with zero lower bound
- 4 exogenous shocks
  - productivity
  - discount factor
  - monetary policy
  - government expenditures.
- Endogenous state: price dispersion

Ferndez-Villaverde et al. (2012): solution method

- Approximation on a sparse grid (Smolyak) in the 5 states
- Time iteration.
- Approximate
  - consumption
  - inflation
  - discounted marginal cost

  as polynomials of the states.
- Use continuous shocks to smooth out the effect of future nonlinearities
- Interest rate R not approximated, but computed from other variables (takes care of kink).

Problem: approximated variables still have a kink, because of contemporary effect of R on output etc.

A Macroeconomic Model with a Financial Sector

Brunnermeier and Sannikov (2012)

- Consumers and experts in production.
- Efficient production requires experts who
  - cannot issue equity
  - require positive net worth
- Strong nonlinear effects through endogenous volatility (leverage of experts).
- Global nonlinear analysis.

Simplifying assumptions

- Continuous time.
- One aggregate shock: depreciation rate of capital
- Output linear in capital, with productivity a for experts and $\underline{a}$ for experts, with $a > \underline{a}$.
- Frictionless market for physical capital
- No idiosyncratic shocks

Solution method

- States: aggregate capital and net worth of entrepreneurs
- Because of linear homogeneity, can be reduced to one:

$$\eta \equiv \frac{\text{Expert net worth}}{\text{Market value of aggregate cap.}} \tag{10}$$

  Follows stochastic differential equation.
- Equilibrium objects (asset price, expert value function, expert leverage) are function of η.
  Satisfy ODE.
- Solution method:
  - Iterate on boundary conditions
  - Solve ODE's.
- Conclusions:
  1. Continuous time makes things easier.
  2. Essential: reduction to one state variable.

Brunnermeier and Sannikov (2012)

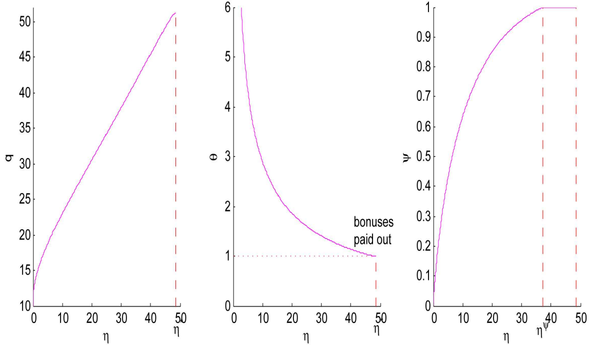

Figure 2: The price of capital, the marginal component of experts' value function and the fraction of capital managed by experts, as functions of η

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- Are difficult to implement.
- Require user choices.
- Can fail to converge.
- Also have problems with kinks.

Recent Advances in Global Methods, I

- Much recent progress for high-dimensional smooth models (Aruoba, Fernandez-Villaverde, and Rubio-Ramirez 2006; Kollmann, Maliar, Malin, and Pichler 2011)
  - Sparse (Smolyak) grids (Barthelmann, Novak, and Ritter 2000; Malin, Krueger, and Kubler 2011; Judd, Maliar, Maliar, and Valero 2013)
  - Efficient computation of expectations, based on Stroud (1971); (Pichler 2011; Fernndez-Villaverde, Gordon, Guerrn-Quintana, and Rubio-Ramrez 2012; Judd, Maliar, and Maliar 2011)
  - Adaptive domain (Judd, Maliar, and Maliar 2012)
  - Perturbation for heterogeneous agent models (Mertens 2011)
  - Projection-perturbation for heterogeneous agent models (Reiter 2010; Reiter 2009)
  - Unertainty shocks:a Krusell-Smith method (Bloom 2009)
- Portfolio choice in GE
  - perturbation, static: Judd and Guu (2000)
  - perturbation, dynamic: Mertens (2011)
  - global solution on event tree: Dumas and Lyasoff (2012)

Recent Advances in Global Methods, II

- Models with occasionally binding constraints (“kinks”)
  - Dynare toolkit by Guerrieri and Iacoviello (2013): certainty-equivalence solutions (as in first-order perturbations)
    Problem: OCB imply strong nonlinearity, which make certainty-equivalence solutions less precise than in smooth models.
  - Precise solution: Extension of endogenous grid point method Carroll (2006) to models with OBC by Hintermaier and Koeniger (2010).
    Problem: may be tedious to apply to higher dimensions.
  - Judd: “get rid of kinks” Judd (2008) ; approximate kinks by smooth (analytic) functions (Kim, Kollmann, and Kim 2010; Mertens and Judd 2011)

Problems with Global (Projection) Methods

- No proof of existence or convergence.
- Computationally expensive: need efficient implementation.
- Difficult to approximate non-smooth functions (variables).

Outline of a Systematic Solution Method

- Solve for deterministic steady state.
- Compute linearized solution around steady state.
- Compute global solution in several steps:
  1. Start with very small shocks; use linearized solution as starting point of nonlinear solution.
  2. In each iteration
     - simulate the model with the most recent solution
     - use simulation results to learn about the state space
     - increase the size of the shocks
     - compute new solution, use last solution as starting point.
  3. Iterate until desired size of shocks is reached.

  This is parallel to the global existence proof in Mertens (2011)!

Other parameters can be adjusted along the path.

Choices

- Which state variables?
- Which domain? (set on which solution lives)
- Which object to approximate?
- Which type of basis functions for approximation?
- How to find fixed point? Newton or time iteration?

Choice of State Variables

- Generally, as few as possible:
  - Portfolio choice without transaction costs: write problem in terms of market wealth of each agent, not of portfolio. (More sophisticated choice: Chien et al. (2011), Dumas and Lyasoff (2012).)
  - Eliminate homogeneity to reduce dimension of state space Example: if all decisions are linearly homogenous in the two states (x, y), one can usually write the whole problem in the variable x/y only. (Example: Brunnermeier and Sannikov (2012).)
- Exception: if more variables allow smoother approximation

Global solution: find the fixed point

- Quasi-Newton methods: solve one big system of nonlinear equations (all residuals at all grid points)
- Time iteration (as in dynamic programming):
  1. Use parameters γ_{t+1} to approximate variables in period $t+1$.
  2. Separately for each grid point: solve equation system for time-t variables.
  3. use time-t variables to update approximation parameters γ_t.
  4. Iterate until convergence.
- Other fixed point iterations are possible (Judd, Maliar, Maliar, and Valero 2013).

Newton vs. Time-Iteration

- Quasi-Newton:
  - quadratic convergence
  - requires solution of system of n linear equations (n is dimension of parameter vector γ).
    - computation is of order n^3
    - memory is of order n^2 (dense case: polynomials; less for splines).
- Time iteration
  - local convergence
  - computation is of order n^2
  - memory is of order n.

For n very large, quasi-Newton may not be feasible.

Domain of Approximation

Set on which the solution lives.

- may be far away from deterministic steady state
- not known ex ante
- must be known to find a solution
- we need iterative procedure
- Iterative approach:
  1. Simulate model, compute covariance matrix of states, use this to determine ellipsoid where state vector is most likely
     Start by region obtained from linearized solution (can be computed analytically for normal shocks)
  2. Cluster grid approach: can handle irregular geometry

Domain of Approximation, ctd.

- Ideally, state space is an ergodic set: if economy is in the set, it stays there with probability 1
  1. Easy in deterministic model
  2. Might be impossible in models with risky assets (even the richest people get richer if the stock market performs very well). Requires extra-polation (rather than inter-polation)! Making state space large relative to size of shocks alleviates instability of extrapolation.
- Use covariance matrix of state to rotate coordinate system
- Unit ball rather than hypercube (ellipsoid rather than rectangle): extreme value of several variables very unlikely.
  Unit ball much smaller than unit cube in high dimensions! Judd (2008).

Approaches to deal with kinks

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  1. If shocks
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     - are big enough (if not, consider certainty-equivalence solution, Guerrieri and Iacoviello (2013) toolkit)

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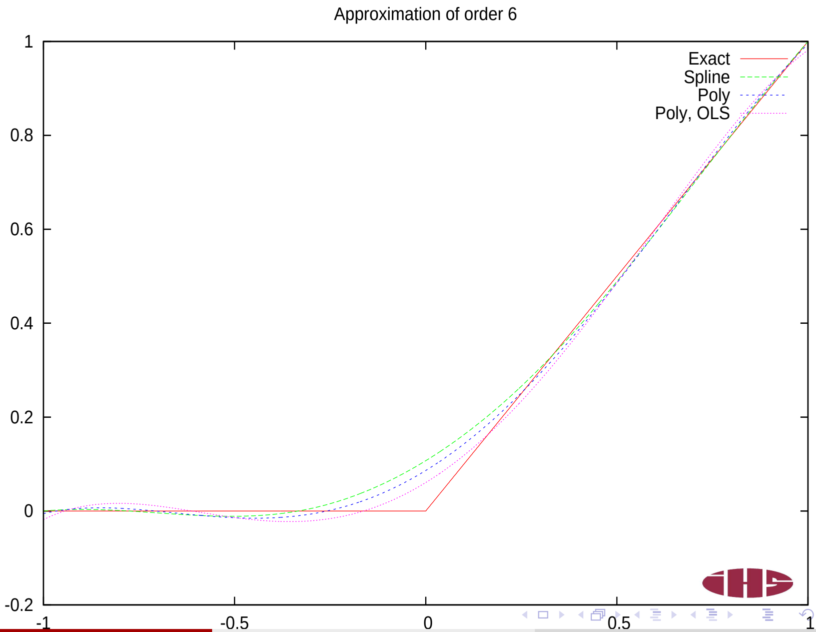
     then approximate expectated values (Wright-Williams Smoothing).
  2. and/or: approximate using more variables, to avoid effect of current non-smooth variables.

Splines

What are splines? Piecewise polynomials.
Example: cubic spline:

- Cubic polynomial between knot points.
- Twice differentiable at knot points.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Exact
Spline
Poly
Poly, OLS
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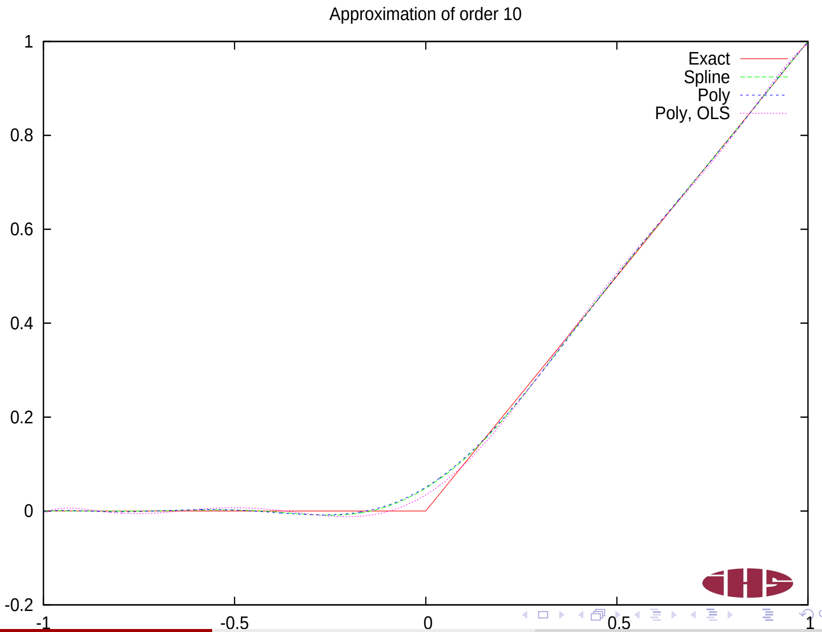
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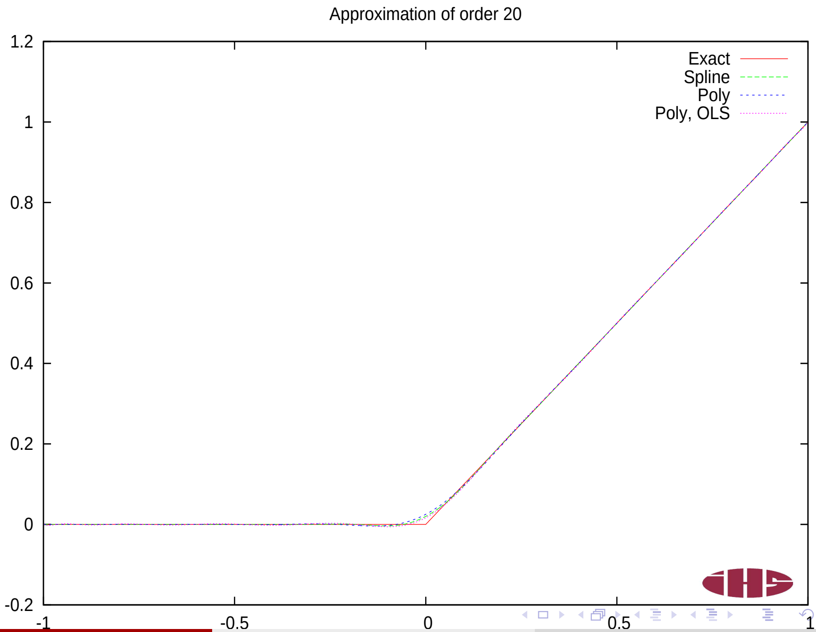
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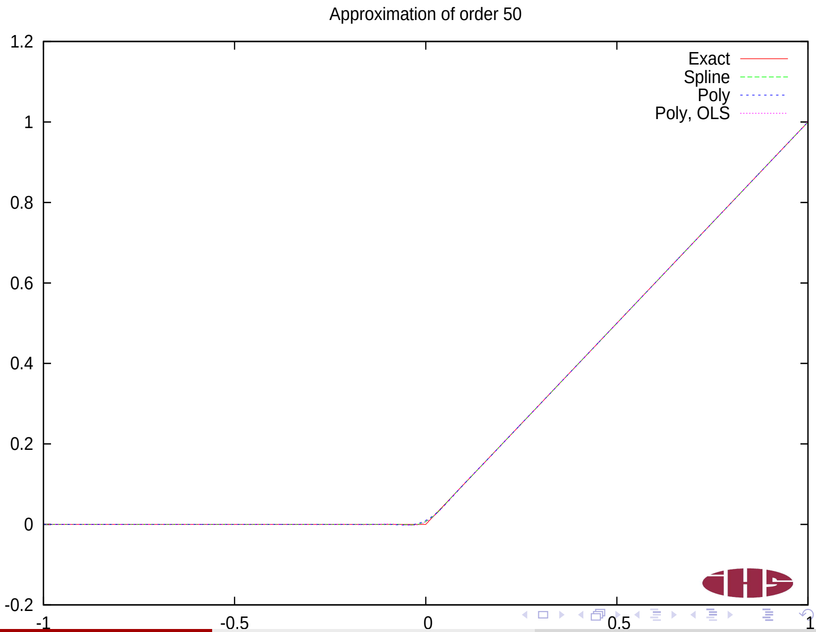
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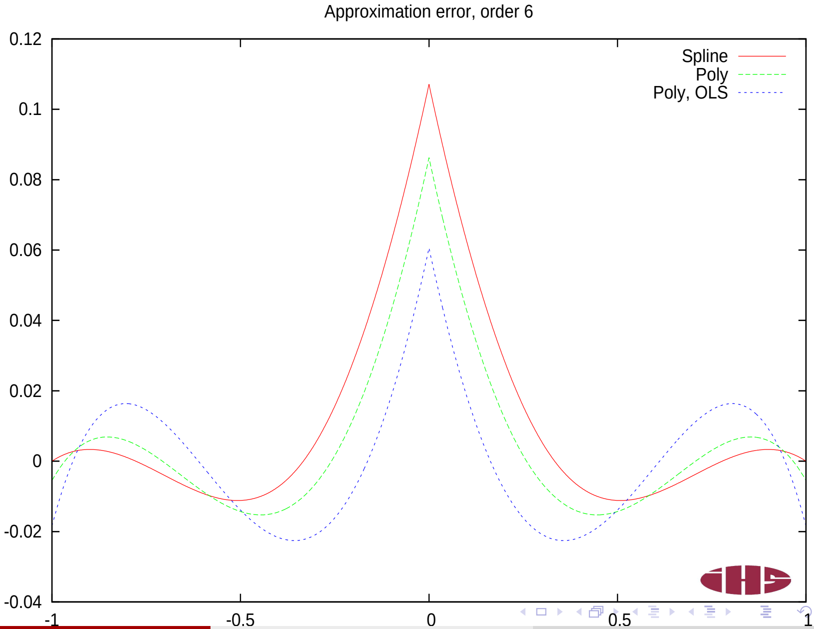
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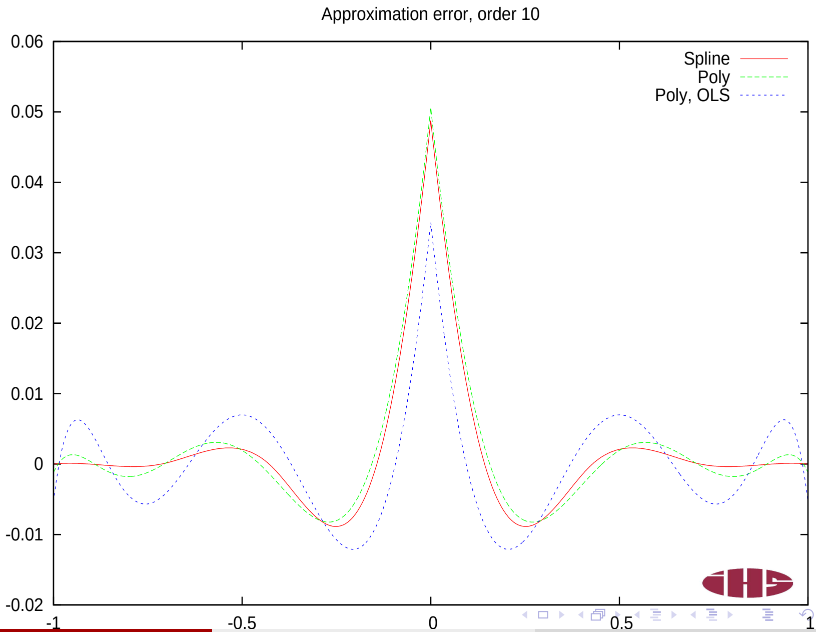
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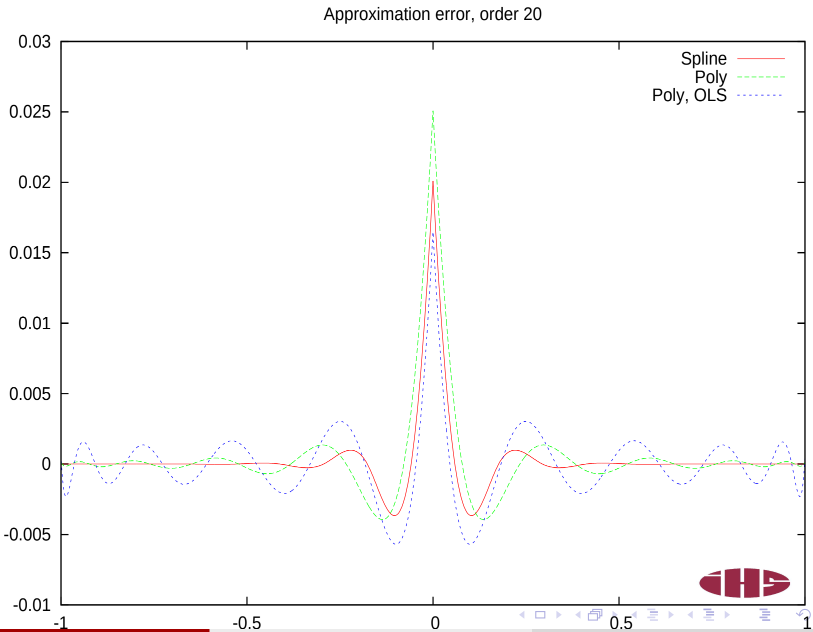
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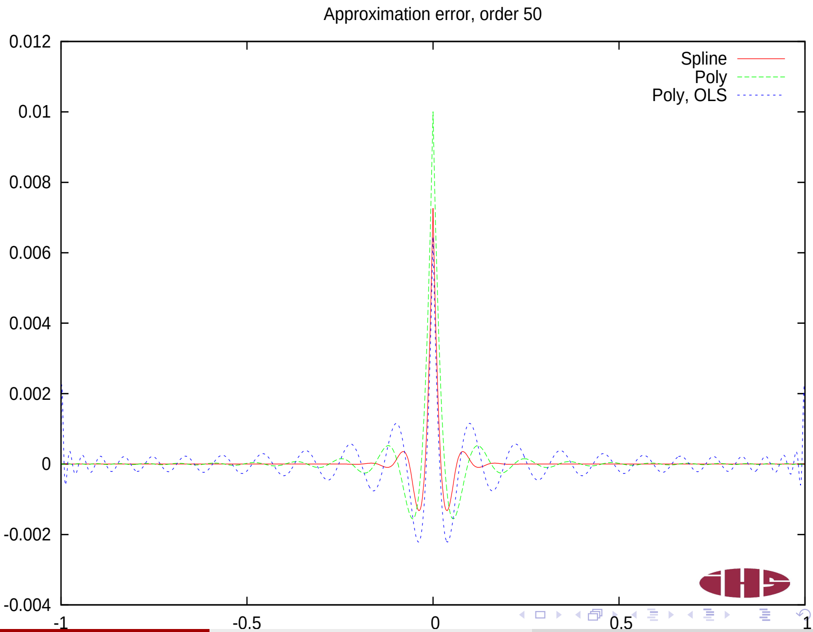
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1

Splines vs. polynomials

1. Splines approximate functions with kinks somewhat better than polynomials with the same number of parameters (degrees of freedom).
2. More importantly: spline approximation can handle more parameters:
   1. Much faster to evaluate (basis functions have local support).
   2. Jacobian matrix (residuals w.r.t. parameters) is sparse.
3. Splines face curse of dimensionality (tensor products).
4. number of parameters of complete polynomials grows polynomially in dimension.

Which Functions (Variables) to Approximate?

Should be smooth!
How can we do this in a model with kinks?

Example 1: asymmetric adjustment costs

Consider a growth model with capital adjustment costs:

$$\max_{k_0,k_1,\ldots} \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t U(c_t) \tag{11a}$$

subject to

$$f(k_t, z_t) = c_t + k_{t+1} - (1-\delta_k)k_t + \Psi(k_{t+1}, k_t) \tag{11b}$$

$$z_{t+1} = \rho z_t + \epsilon_{t+1}, \qquad \mathrm{E}_t\, \epsilon_{t+1} = 0 \tag{11c}$$

$$\tag{11d}$$

where we assume

$$\Psi(k', k) = \begin{cases} \frac{1}{2k}\underline{\psi}(k'-k)^2 & \text{if } k' \geq k \\ \frac{1}{2k}\overline{\psi}(k'-k)^2 & \text{if } k' < k, \end{cases} \qquad \text{with } \overline{\psi} > \underline{\psi} \tag{12}$$

Then the optimal investment function $k'(k, z)$ has a kink at any point (k, z) where $k' = k$.

Parameterizing Expectations

The Euler equation is

$$U'(c_t)\,(1+\Psi_1(k_{t+1},k_t)) = \beta\,\mathrm{E}_t\left[U'(c_{t+1})\,(1+f_k(k_{t+1},z_{t+1})-\Psi_2(k_{t+2},k_{t+1}))\right] \quad (13)$$

k_{t+2} and c_{t+1} are functions of (k_{t+1}, z_{t+1}). Define

$$\Gamma(k_{t+1},z_{t+1}) \equiv U'(c_{t+1})\left(1+e^{z_{t+1}}f'(k_{t+1})-\Psi_2(k_{t+2},k_{t+1})\right) \quad (14)$$

1. $\Gamma(k_{t+1},z_{t+1})$ has a kink where $k_{t+2}=k_{t+1}$
2. If ϵ has a smooth density $\xi(\epsilon)$, then $\mathcal{E}^{WW}$ is smooth:

$$\mathcal{E}^{WW}(k_{t+1},z_t) \equiv \int \Gamma(k_{t+1},\rho z_t+\epsilon)\xi(\epsilon)\,d\epsilon \quad (15)$$

3. But not

$$\mathcal{E}^{PE}(k_t,z_t) \equiv \mathcal{E}^{WW}(k_{t+1}(k_t,z_t),z_t) \quad (16)$$

Wright-Williams Smoothing

- Approximating $\mathcal{E}^{WW}(k_{t+1}, z_t)$ is called "Wright-Williams smoothing" (Judd 1998, p.586ff.).
- In parameterized expectations algorithm, usually $\mathcal{E}^{PE}(k_t, z_t)$ is approximated, which is not a smooth (differentiable) function (pointed out by K. Judd, but often ignored!)

Example 2: NK model with ZLB

Basic NK model:

$$y_t = \mathrm{E}_t(y_{t+1}) - 1/\sigma(R_t - \mathrm{E}_t(\pi_{t+1})) + z_{y,t} \tag{17a}$$

$$\pi_t = \beta\,\mathrm{E}_t(\pi_{t+1}) + \lambda mc_t \tag{17b}$$

$$mc_t = \kappa y_t + z_{mc,t} \tag{17c}$$

Taylor rule:

$$\hat{R}_t = \phi_p \pi_t + \phi_y y_t \tag{17d}$$

ZLB (0.005 is StSt inflation):

$$R_t = \max(\hat{R}_t, -0.005) \tag{17e}$$

Shocks:

$$z_{mc,t} = \rho z_{mc,t-1} + \epsilon_{mc,t} \tag{18a}$$

$$z_{y,t} = \rho z_{y,t-1} + \epsilon_{y,t} \tag{18b}$$

Approximation I: WW smoothing

- No endogenous state: if shock is smooth, then $E_t(y_{t+1})$ and $E_t(\pi_{t+1})$ are smooth variables.
  Approximate them as functions of the exogenous states z_{mu} and z_y.
- Given $E_t(y_{t+1})$ and $E_t(\pi_{t+1})$, solve equation system (17) for y, π, $\hat{R}$, R and mc.

Is this enough?
Works reasonably well if $E_t(y_{t+1})$ and $E_t(\pi_{t+1})$ are approximated by splines, not polynomials.

Approximation II: R as auxiliary state

- Approximate y_t and π_t as functions of $z_{y,t}$, $z_{mu,t}$ and R_t.
- Requires again to solve an inner loop:
  1. Guess R_t
  2. Evaluate y_t and π_t
  3. Check guess of R_t.

Works well, even with polynomial approximations!

NK Model, Parameters

discount factor: $\beta = 0.96^{1/4}$
inverse LSE: $\phi = 1$
risk aversion; $\sigma = 1$
demand elasticity; $\epsilon = 7$
Taylor rule, output; $\phi_y = 0.125$
Taylor rule, inflation; $\phi_p = 2.5$
persistence of shocks; $\rho_z = 0.5$

$\alpha = 0.3$
$\Theta = (1-\alpha)/(1-\alpha(1-\epsilon))$
$\theta = 0.6667$
$\lambda = (1-\theta)(1-\beta\theta)/\theta\Theta$
$\kappa = \sigma(\phi+\alpha)/(1-\alpha)$

NK Model, Size of Shocks

- Shock to marginal cost: uniformly distributed on $[-0.035, 0.035]$.
- Shock to Euler equation: uniformly distributed on $[-0.007, 0.007]$.
- ZLB binding in about 10% of periods.
- With larger shocks: solution fails to converge!

Approximation errors, Output

	Spline 10	Spline 20	WW Sm., 10	WW Sm., 20
Mean	7.29e-6	-6.38e-7	7.74e-7	9.05e-7
Mean abs.	1.17e-4	3.43e-5	1.51e-6	1.57e-6
Max abs.	6.15e-4	3.47e-4	1.00e-5	9.21e-6

Errors for Approximation II

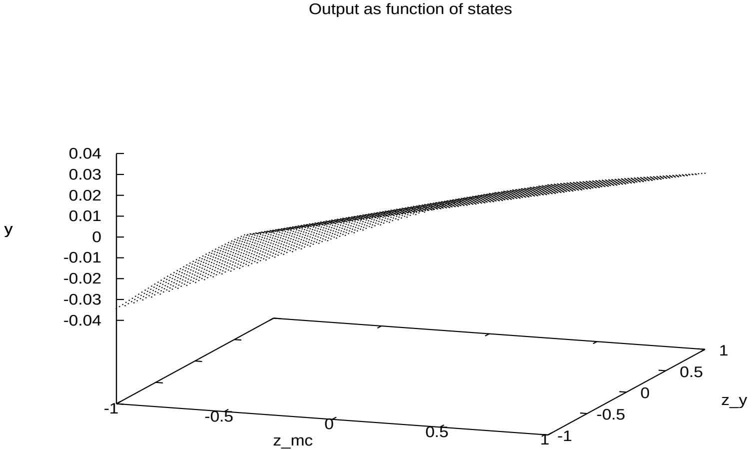
Approximation:

- cubic polynomial in y, π
- linear in R

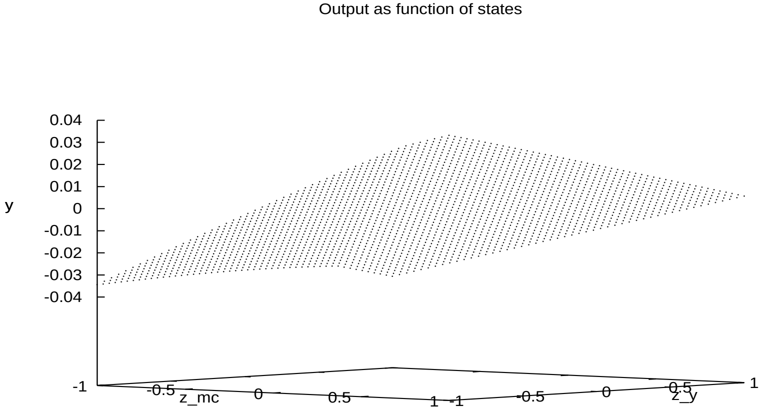
Residuals from 2500 random points in state space:

	mean	mean(abs)	max(abs)	max(abs) rel.
Output:	7.23e-08	1.01e-06	3.28e-06	1.82e-04
Inflation:	1.19e-08	1.52e-07	5.06e-07	1.11e-04

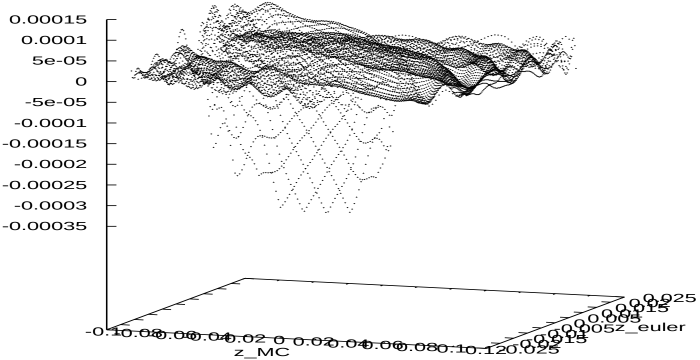
Output y


Output as function of states
y
0.04
0.03
0.02
0.01
0
-0.01
-0.02
-0.03
-0.04
-1
-0.5
0
0.5
1
z_mc
-1
-0.5
0
0.5
1
z_y

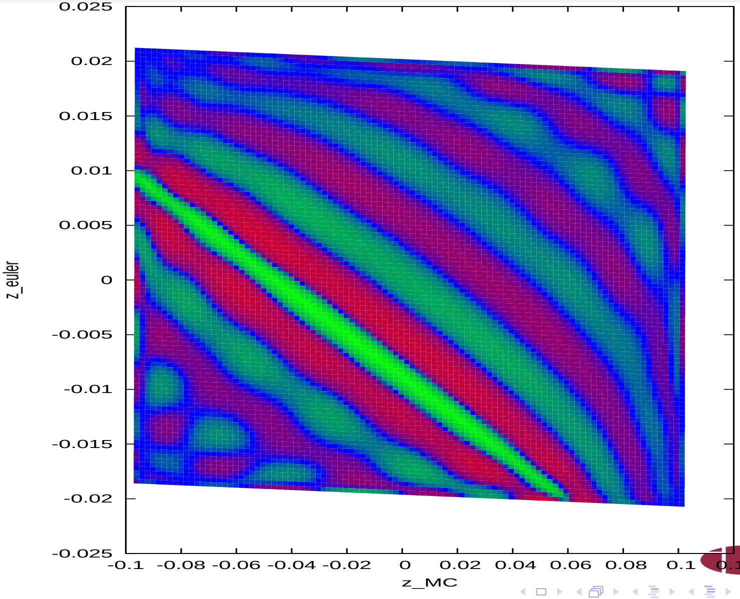
Output


Output as function of states
0.04
0.03
0.02
0.01
y
0
-0.01
-0.02
-0.03
-0.04
-1
-0.5
z_mc
0
0.5
1
-1
-0.5
0
0.5
z_y
1

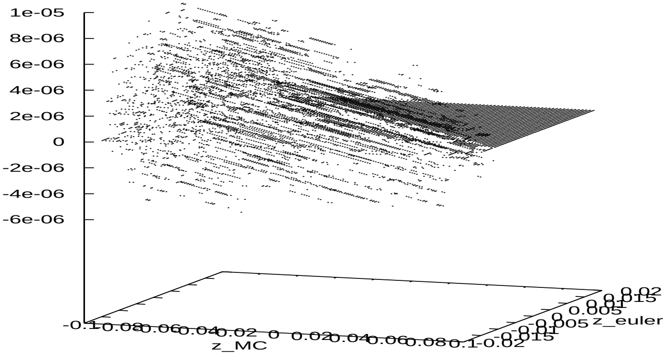
SError y, pline order 20



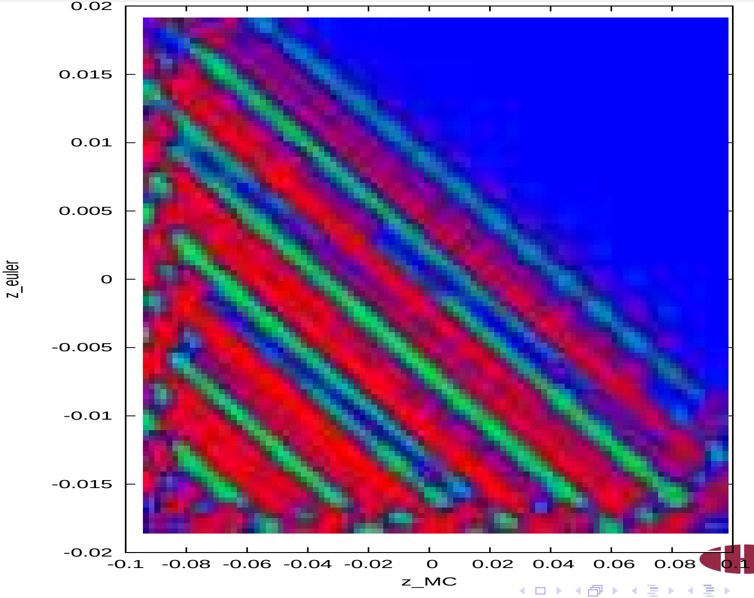
Error y, Spline order 20



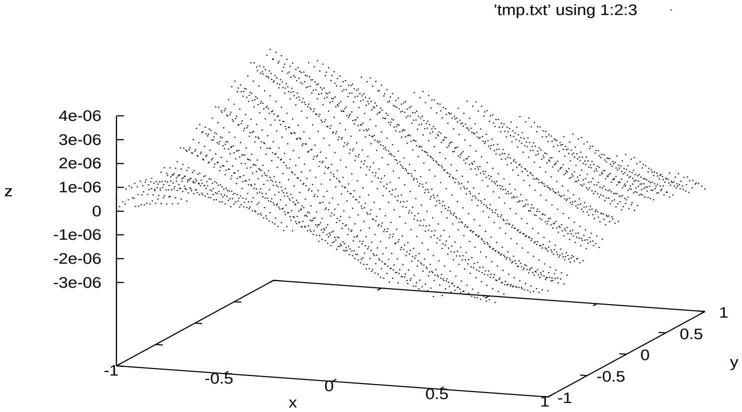
Error y, Spline order 20,WW Smoothing



Error y, Spline order 20,WW Smoothing



Error y, Polyn. in R



Necessary: Two-Step Approximation

1. Approximated object: set of smooth functions A (x)
2. Decision $y(x)$ is obtained from A (x) by solving

$$G(A(x), y(x)) = 0 \tag{19}$$

Efficient implementation: exploit

$$\frac{\partial y}{\partial \gamma} = \frac{\partial y}{\partial A}\frac{\partial A}{\partial \gamma} = -G_y^{-1}(A(x), y(x))G_A(A(x), y(x))\frac{\partial A}{\partial \gamma} \tag{20}$$

Compute by **Automatic Differentiation.**

Automatic Differentiation

- Numerical computation of derivatives of a function $f(x)$ at a specific point x_0, using the exact rules of differentiation.
- Example:

$$f(x) = log(x \cdot sin(x^3)); \tag{21}$$

$$a = x^3; b = sin(a); c = x \cdot b; f(x) = log(c)$$

Then, at $x = x_0$,

$$da/dx = 3 \cdot x_0^2 \tag{22a}$$
$$db/dx = cos(a) \cdot da/dx \tag{22b}$$
$$dc/dx = b + x_0 \cdot db/dx \tag{22c}$$
$$df/dx = (1/c) \cdot dc/dx \tag{22d}$$

These computations can be automatized by the computer in object-oriented programminj through operator overloading.

Efficiency of Automatic Differentiation

1. Theoretical result: computing the complete gradient of a function $f : \Re \rightarrow \Re^n$ takes not more than 5 times the operation that it takes to compute f.
   But this is difficult to implement (reverse mode).
2. The chain of calculations in (22) is easy to implement, but often rather inefficient.
3. In our application, even forward mode rather efficient, because
   - use of implicit function theorem in two-step approximation
   - function evaluation:
     $$\frac{\partial A(x)}{\partial \gamma} = \frac{\partial A(x)}{\partial x} \frac{\partial x}{\partial \gamma} \tag{23}$$
     Since x has much fewer elements than γ, (23) is much faster than computing $\frac{\partial A(x)}{\partial \gamma}$ by forward differentiation!

Example: 3-period OLG with Portfolio Choice

Technology

$$y_t = F(K_t, L_t) \tag{24}$$

$$y_t = K_{t+1} - (1 - \delta_t K_t + C_t \tag{25}$$

$$r_t = F_k(K_t, L_t) - \delta_t \tag{26}$$

$$w_t = F_L(K_t, L_t) \tag{27}$$

Stochastic depreciation factor:

$$\delta_t = \bar{\delta} + \rho(\delta_{t-1} - \bar{\delta}) + \epsilon_t \tag{28}$$

Model: Assets

Capital K and riskless bond A

$$K_t = ((W_{2,t} + W_{3,t})/(1 + r_t)) \tag{29}$$

$$K_{1,t} = (w_t \zeta_1 L_1 - c_{1,t} - q_t^S A_{1,t}) \tag{30}$$

$$W_{2,t} = K_{1,t-1} R_t^K + A_{1,t-1} \tag{31}$$

$$K_{2,t} = (W_{2,t} + w_t \zeta_2 L_2 - c_{2,t} - q_t^S A_{2,t}) \tag{32}$$

$$W_{3,t} = K_{2,t-1} R_t^K + A_{2,t-1} \tag{33}$$

Euler Equations

$$\beta\,\mathrm{E}_t(R^K_{t+1}U_c(c_{2,t+1},L_2)) = U_c(c_{1,t},L_1))$$
$$\beta\,\mathrm{E}_t(U_c(c_{2,t+1},L_2)) = q^S_t U_c(c_{1,t},L_1)) + \kappa A_{1,t}$$
$$\beta\,\mathrm{E}_t(R^K_{t+1}U_c(c_{3,t+1},0)) = U_c(c_{2,t},L_2))$$
$$\beta\,\mathrm{E}_t(U_c(c_{3,t+1},0)) = q^S_t U_c(c_{2,t},L_2))$$

$\kappa \neq 0$ forces $A_{i,t} = 0$ in steady state and linear approx.
Short-sale and collateral constraints:

$$K_{i,t} \geq 0$$
$$A_{i,t} + \phi K_{i,t} \geq 0 \tag{34}$$

Asset market equilibrium:

$$A_{1,t} + A_{1,t} = 0; \tag{35}$$

Model: Aggregation

$$c_{3,t} = W_{3,t} \tag{36}$$

$$L_t = \zeta_1 L_1 + \zeta_2 L_2 \tag{37}$$

$$C_t = c_{1,t} + c_{2,t} + c_{3,t} \tag{38}$$

Solving the Portfolio Choice Model: Outline

- State variables: Household net worth of each cohort at beginning of period (assuming no trading frictions)
  Not: whole asset position (too many states).
- Approximate: consumption of each cohort.
  Not: asset choice. (because of kinks: short-sale constraints)
- Given time-$(t+1)$ consumption function, calculate asset position in period t from Euler equations.

Problems with portfolio choice model

- Asset choice indeterminate in steady state and in linearized solution
  Solution:
  - punish holdings of safe asset by parameter κ
  - set $\kappa = 0$ in final step of nonlinear solution.
- Difficult to find compact stable state space (extreme realizations of asset returns kick households beyound bounds).
  (Partial) Solution: state space large compared to shocks $\Longrightarrow$ mild extrapolation.

Problems with portfolio choice model, ctd.

- State variables of time t (market value of wealth) depend on endogenous variables of time t (asset returns).
  Solution: iterate over portfolios and over interest rate in each computation of state transition function.
  1. Select asset returns as variables that are approximated as function of states
  2. At t, guess
     - portfolio choices of each cohort
     - asset returns (here: interest rate) of $t+1$ for each possible realization of exogenous shocks $t+1$
  3. Compute next period's wealth levels
  4. Update guesses until
     - Euler equations are satisfied
     - Guessed asset returns consistent with approximation.

Accuracy portfolio choice model

Mean(abs) error:

	K/L	C_1	C_2
3:	3.9322e-04	4.0989e-04	4.0057e-04
4:	1.8495e-05	1.7580e-05	1.8194e-05
6:	2.3788e-06	2.3607e-06	2.4118e-06

Max(abs) error:

	K/L	C_1	C_2
3:	1.3640e-03	1.2223e-03	1.5433e-03
4:	6.0046e-04	5.1620e-04	6.8820e-04
6:	1.0832e-04	9.0181e-05	1.1168e-04

A Toolkit

- Linear part
  - Similar to Dynare
  - Systematic way to compute steady state
  - Can handle large models through loop constructions.
- Nonlinear part
  - Projection method
  - Choice of different approximation schemes
    - Complete polynomials
    - Splines
    - Sparse grids
    - Combinations (tensor products) of these.
- Implementation
  - Written in C++
  - Each project gets compiled in C++ (but the user need not know any C++)
  - Output in Matlab form, so it can be analyzed in Matlab.

Toolkit file NK, linear part

```
z(0) := rho_z*z(-1) + eps_mc;
z2(0) := rho_z*z2(-1) + eps_euler;
y(0) := EXP(y(1)) - 1/sigma * (IntR(0)-EXP(pi(1))) + z
pi(0) := beta*EXP(pi(1)) + lambda*mc(0);
mc(0) := kappa*y(0) + z(0);
R0(0) := phi_p * pi(0) + phi_y *y(0);
IntR(0) := max(R0(0),-0.005);

STST;
@LINEARMODEL;
END;
```

Toolkit file NK, nonlinear part

```
STATE(z(0),z2(0));
APPROX(y,pi);
DEFINE(mc,R0,IntR);

REALIZ(eps_mc;-0.03:1/3, 0:1/3, 0.03:1/3);
REALIZ(eps_euler;-0.005:1/2, 0.005:1/2);
STEPSSIGMA(0.01,20);
PROBSPACE(0.999);
DOSPLINE(20,20);
```

Toolkit file NK, WW smoothing, nonlinear part

```
STATE(z_mc(0),z_y(0));
APPROX(Ey,Epi);
GUESSSTATIC(IntR);
DEFINE(y,mc,pi,R0);

REALIZ(eps_mc;-0.03:1/3, 0:1/3, 0.03:1/3);
REALIZ(eps_y;-0.005:1/2, 0.005:1/2);
STEPSSIGMA(0.01,20);
PROBSPACE(0.999);
DOSPLINE(20,20);
```

Conclusions

- Systematic procedure, starting from linearization around stationary state
- Many implementation issues,
  - Fast computation of polynomial and spline approximations, sparse grids.
  - Automatic differentiation, dense and sparse Jacobian
  - Parallelization

  Toolkit needed.
- TODO:
  - Discretionary policy (dynamic games)
    Problem: steady state is not a system of algebraic equations.
    - Homotopy from full commitment to no-commitment through “loose commitment” Debortoli and Nunes (2010)
    - Iterate over derivatives at StSt.
  - Nonlinear solution of models with continuum of agents.

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