Solving for Optimal Portfolio Dynamic Choices with Multiple Agents and Multiple Assets: An asymptotic approach with some open economy applications*
(PRELIMINARY AND INCOMPLETE)

Luca Dedola — ECB and CEPR

Giovanni Lombardo — ECB

August 21, 2009

Abstract

In this paper we present methods to solve for asset allocations in dynamic incomplete market economies, in which the impact of portfolio choices goes beyond the cases studied in the perturbation literature so far (e.g. Devereux and Sutherland (2008, 2007a), Van Wincoop and Tille (2007) and Judd and Guu (2001)). For instance, this more general case is relevant in characterizing Ramsey optimal policies with multiple agents and assets under incomplete markets. We also clarify the link between the Devereux-Sutherland solution methods and the asymptotic approach proposed by Judd and Guu to deal with bifurcations arising in static portfolio problems. Finally, we extend the solution technique proposed by Devereux and Sutherland, by allowing more than two agents (and multiple assets) in dynamic incomplete markets economies.

We present three open-economy applications of our methods. First, we solve the portfolio problem in a simple three-country three-bond nominal economy for which we can find analytical results for the steady-state portfolio. This solution shows that relative risk can have compositional effects on the portfolio that would not exist in a two-country model, with potentially interesting implications for the study of the interactions between optimal exchange rate regimes and portfolio allocation. In our second application, we solve for the country portfolios dynamics in response to shocks in a simple two-country model with both equities and nominal bonds, studying how home equity bias reacts to shocks. In our third and last application we solve for the optimal nominal bond portfolio under Ramsey monetary policy in a canonical 2-country model with

*The views expressed in this paper do not necessarily represent the view of the European Central Bank.

Calvo pricing, and technology and mark-up shocks, with both complete and incomplete markets.

Contents

1 Introduction

Analyses of asset markets equilibrium in dynamic incomplete market economies are very difficult, and there are very few cases that can be solved exactly for equilibrium prices and quantities. The difficulty arises from the fact that portfolio choices are indeterminate in the absence of uncertainty. Since standard methods to solve for dynamic economies use the nonstochastic steady state as the starting point of approximation, they cannot be readily applied to these problems.

In two important recent papers, Devereux and Sutherland have derived the optimal portfolio composition for dynamic macro models with two agents (Devereux and Sutherland, 2007a, 2008). They show that using standard first-order solution techniques it is possible to determine the "near-stochastic" optimal portfolio allocation around which the non-linear dynamic model can be approximated. Furthermore they show that using simple second order approximation techniques, it is possible to characterize the dynamics of this portfolio, up to first order of accuracy. Van Wincoop and Tille (2007) propose iterative techniques to solve for the optimal portfolio based on the same principles inspiring the work of Devereux and Sutherland.

In this paper we extend their results along several dimensions. First, we extend these methods to solve for dynamic problems in which portfolio allocations are still indeterminate in the nonstochastic steady state, but relaxing the assumption that these allocations only appear multiplied by excess returns. The latter assumption is necessary for the solution technique suggested by Devereux and Sutherland and by Tille and van Wincoop.[1] The more general case could be shown to be relevant in a number of important problems, for instance if one is interested in solving for Ramsey optimal policies with multiple agents and assets under incomplete markets. The Devereux and Sutherland solution does not cover this class of interesting problems. We show how to solve this more general class of models using iterative methods, providing an application to the optimal nominal bond portfolio under Ramsey monetary policy in a canonical 2-country economy with Calvo pricing and technology and mark-up shocks.

In deriving these results, we provide a further contribution by clarifying the link between the Devereux-Sutherland methodology and the asymptotic approach proposed by Judd and Guu (2001), in which bifurcation techniques (BT) are used to address the failure of the implicit function theorem (IFT) when perturbations methods are applied to approximate the solutions to static portfolio problems with small risks. We show that the two approaches share the same formal foundations. Importantly, also in the Judd-Guu class of problems, portfolio allocations only appear multiplied by excess returns.

Finally, we derive closed form solutions for the class of economies studied by Devereux and Sutherland in the case of more than two agents, and also more than two assets, for the dynamics. We derive these solutions with a relatively

[1] For example "The only two ways that portfolio shares enter model equations are (i) through the return on the overall portfolio and (ii) through asset demand." Assumption 1 in Van Wincoop and Tille (2007).

compact matrix algebra, which should facilitate the generation of computer codes. A number of important questions can be properly addressed only with a multi-country model, especially concerning the links between financial globalization and monetary policy. For example, a large literature has addressed the question of optimal exchange rate regimes and optimum currency areas. The typical trade-off emphasized by this literature is that between independently choosing imperfect stabilization policies and gaining credibility by pegging the exchange rate to the currency of a better managed economy (e.g. Giavazzi and Pagano (1988), Ravenna (2005) and Clerc et al. (2008)). Neumeyer (1998) has showed that eliminating currencies can amount to reducing assets (if nominal assets were available) and could, therefore, reduce the amount of risk sharing among countries. While in a two-country model choosing to peg the exchange rate amounts to eliminating all possibilities to hedge risk by holding foreign currency nominal bonds, in a multi-country model this need not be the case, as long as some currencies remain independent. Furthermore, the hedging role of the exchange rate could generate strategic motives in deciding the admission of new members in a currency union or in deciding whether to join an existing currency union.

Using our solutions for optimal steady-state and dynamic portfolio allocations for multiple agents and assets, we analyze two simple models. The first is a three-country, three-bond endowment economy model, with monetary and endowment shocks. We show that having more than two agents can generate portfolio compositions that are ruled out in the two-country setup. For example, as the variance of the monetary policy shocks of one country becomes infinitely large, all countries will still hold a nontrivial portfolio of bonds in the other two currencies. On the contrary, in the two-country model the optimal portfolio would display zero holding of all bonds. This result is potentially relevant for the literature analyzing the interactions between optimal exchange rate regimes under incomplete markets, as it suggests that two countries might fix their bilateral exchange rate while allowing a float with respect to other third countries, thereby preserving the diversification opportunities provided by a sufficiently large set of nominal assets.

The second application consists of a two-country model with trade in equities and bonds and with money, endowment and dividend shocks. With this model we can address the question of equity home bias. We show that the long-run equity position is affected by the relative persistence of dividend and endowment shocks. In the special case of equal persistence of dividend and endowment shocks, and without monetary shocks, we can produce perfect equity home-bias (e.g. as discussed in Coeurdacier et al. (2007)). In addition, we also study the dynamic responses of equity and bond holdings in response to shocks, with both complete and incomplete markets.

In the complete-market case the wealth distribution is stationary. In general, though, net-wealth (and real allocations) is not stationary when the model is approximated around the non-stochastic steady state under incomplete markets. In principle this could obscure the interpretation of the long-run portfolio compositions. Nevertheless, we show that introducing a stochastic discount factor, as discussed in Schmitt-Grohé and Uribe (2003), we can eliminate the non-stationarity without altering the main results, for reasonable parameterization of the discount factor.

Finally, the third application shows that the solution proposed by Devereux and Sutherland (2008, 2007a) cannot be applied to an interesting class of models, including models with Ramsey policymakers. We show that the general solution technique, nevertheless, allows us to characterize the portfolio also in this broader class of models. We apply the general solution to the canonical two-country model with sticky prices used among others by Benigno and Benigno (2006). We show that when there are only productivity shocks, PPI-stability coincides with the optimal Ramsey monetary policy also in terms of optimal portfolio allocation. In the presence of mark-up shocks, the Ramsey policy induces a different portfolio allocation than the one obtained under the PPI-stability policy.

Other papers discuss the derivation of optimal portfolios in open-economy models. Coeurdacier et al. (2008) find a closed form solution for a two country model with trade in stocks and bonds. Their analysis is close to the work of Heathcote and Perri (2007) by showing that equity home bias can be the result of optimal hedging of idiosyncratic risk. Their derivation of the portfolio solution is based on the assumption of complete markets and, therefore, differs from ours.

Van Wincoop and Tille (2007) propose a solution method for optimal portfolios that is equivalent to that discussed in Devereux and Sutherland (2007a, 2008). Our derivation of the closed form solution differs substantially from theirs. Different is also the derivation of the optimal portfolio obtained by Evans and Hnatkovska (2007). These authors also apply approximation methods to compute the optimal (dynamic) portfolio in DSGE models with multiple assets. Nevertheless they combine discrete-time perturbation methods with continuous-time approximation methods in order to characterize the portfolio.[2]

Engel and Matsumoto (2006) and Engel and Matsumoto (2008) in particular, show that price stickiness is an important determinant of the portfolio composition. Considering different assumptions regarding the currency used in setting prices (i.e. local currency vs. producers' currency) they show (analytically) that exchange rate risk is the most important determinant of the portfolio composition when prices are sticky. In this context only a small trade in equities is necessary alongside trade in bonds to replicate complete markets.

The rest of the paper is organized as follows. Section 2 defines the reference model that we are going to solve.[3] Section 3 suggests an algorithm to solve for the optimal portfolio when the conditions necessary for Devereux-Sutherland method do not apply. Section 4 compares the asymptotic solution method of Judd and Guu with that proposed by Devereux and Sutherland. Section 5 derives the optimal portfolio with multiple agents and multiple assets in the near-stochastic steady state. Section 6 derives the optimal portfolio dynamics.

[2] A continuous-time two-country model is used by Pavlova and Rigobon (2008) to solve the portfolio problem. These authors stress the importance of including optimal portfolio decisions in open economy models in order to understand the dynamics of the current account.

[3] A more general model will be discussed further below.

The following two sections discuss applications of our formulae. In particular, Section 7 derives the optimal bond portfolio in a three-country endowment economy. Section 8 derives the optimal equity and bond portfolio in a two-country endowment economy. Section 9 applies the general solution technique to a two-country model with sticky prices and Ramsey optimal monetary policy. Section 10 concludes.

2 The reference model and the indeterminacy of the portfolio

Let's assume, for the sake of concreteness, that there are n countries and k assets internationally traded.[4] Each country is populated by a representative household with the following utility function (here for country j)

$$U_t = E_t \sum_{\tau=t}^{\infty} \beta^{\tau-t} \left[u\left(C_{j,\tau}\right) + v\left(\cdot\right) \right], \tag{1}$$

where C is consumption and $v(\cdot)$ refers to terms not relevant for our analysis.

The budget constraint agent j is of the type

$$W_{j,t} = \sum_{i=1}^{k-1} \alpha_{i,t-1}^{j} \left(r_{i,t}^{j} - r_{B,t}^{j} \right) + \left(Y_{j,t}\right) + r_{B,t}^{j} W_{j,t-1} - C_{j,\tau}, \tag{2}$$

where, following Devereux and Sutherland (2008), $\alpha_{i,t}$ denotes the quantity of the particular asset i and $W_{j,t}$ denotes the net asset holding so that $\sum_{i=1}^{k} \alpha_{i,t}^{j} = W_{j,t}$. We denote by $r_{i,t}^{j}$ the real return on the asset i and by $r_{B,t}^{j}$ the return on the reference asset, in terms of the consumption basket of agent j, so that $r_{x,i,t}^{j} \equiv \left(r_{i,t}^{j} - r_{B,t}^{j} \right)$ measures the excess return of asset i relative to the reference asset B. Y_t measures output.

The first order conditions for the choice of the assets can be written as

$$E_t \left[u'\left(C_{t+1}^{j} \right) r_{i,t+1}^{j} \right] = E_t \left[u'\left(C_{t+1}^{j} \right) r_{B,t+1}^{j} \right]; \; i = 1 \ldots k-1 \text{ and } j = 1 \ldots n. \tag{3}$$

For concreteness assume that the economies are subject to e independent (but possibly serially correlated) shocks. We denote the innovations to this shocks with the $e \times 1$ vector ε_t.

The set of equations (3) describe the condition governing the optimal choice of portfolio. In a non-stochastic equilibrium all these conditions imply that all

[4]Notice that n could be larger or smaller than k. For example, a multi-country model with only nominal bonds would admit as many distinct assets as independent currencies.

real return are identical. Therefore, agents are indifferent about the composition of their portfolio in a world without risk. Likewise, to a first order of approximation, certainty equivalence holds and these conditions imply that ex-ante all returns are identical. Therefore, agents are indifferent about the composition of their portfolio in a world of certainty equivalence. Only when certainty equivalence does not hold, i.e. at higher orders of approximation, equations (3) provide conditions to determine the demand for assets.

This indeterminacy of the portfolio constitutes a serious problem for the solution of DSGE models using standard perturbation methods. These methods, generally require us to take an approximation around the non-stochastic steady state. A point in which the portfolio is indeterminate.

Judd (1998) and Judd and Guu (2001) have suggested asymptotic methods to address portfolio indeterminacy in a static setting, while Devereux and Sutherland (2008) have developed a convenient technique to derive the solution for a large class of intertemporal models. These two approaches are compared in Section 4.

Before discussing the details of these techniques it is important noticing that equations (3) can be used to assess whether any proposed portfolio allocation is indeed optimal from the agent's point of view. In the following section we exploit this fact in proposing a solution for a larger class of models than those studied by Devereux and Sutherland (2008) or by Van Wincoop and Tille (2007).

3 Finding a solution when the portfolio enters the model in a more general form

In this section we discuss a more general solution for the optimal steady-state portfolio when portfolio shares do not enter the model only as multiplied by the excess return, but are still indeterminate in the non-stochastic steady state. The property that the portfolio only multiplies excess returns is a necessary condition for the application of the solution method suggested by Devereux and Sutherland (2008) and by Van Wincoop and Tille (2007). The condition guarantees that i) to first-order of approximation of the system the portfolio enters the model only as a constant and, ii) one can conflate this constant in the auxiliary i.i.d. term in the Devereux and Sutherland (2008) solution technique.[5]

In general, when the portfolio produces externalities, it might not be possible to satisfy conditions i) or ii) or both. For example, consider the solution to the optimal Ramsey monetary policy problem. The portfolio choice will affect the policymaker FOCs in a way that violates condition i) or ii).

[5]The contribution of these papers is not limited to the proposed solution technique. They also highlight the point that simple approximation methods can be used to evaluate the moment conditions derived from the optimal portfolio choice. Neither of these papers, though, hints at the fact that simple approximation methods can be used to solve the portfolio problem in a larger class of models. See next section for further details.

Economic models with standard portfolio problems produce singular Jacobian matrices (Judd and Guu, 2001). This is due to the fact that each agents' FOC for the choice of a particular asset is not linearly independent relative to the FOC of the other agents: to first order (i.e. linearly) they all impose that expected excess returns be zero. Judd and Guu (2001), show that in this case one can still use perturbation methods and resort to the bifurcation theorem to find a solution to the portfolio problem. In essence this amounts to finding higher-order approximations of the optimality conditions for which the portfolio choice ceases to be indeterminate. The optimal portfolio will then be the portfolio that satisfies those optimality conditions at the appropriate level of approximation.

This is the case for equations 3, describing the optimal choice of portfolio shares. In particular, if we have k assets and n agents we must exclude $k-1 \times n-1$ equations. So, for example, with two assets and two agents we can only include the FOC for the choice of one portfolio share (one asset) for only one of the agents. Notice also that the FOC for the choice of net-wealth (the whole portfolio) is included for each of the agents.[6]

The FOC for the choice of a particular asset i and agent j are of the form

$$E_t U'(C^j_{t+1}) r_{x,t+1} = 0 \tag{4}$$

We have argued that $k-1 \times n-1$ of these equations are linearly dependent. In what follows we will refer to these conditions as moment conditions.

For clarity and comparability we distinguish two cases: First, the case in which the portfolio enters the model multiplied by variables that are zero in the non-stochastic steady-state, whether they are i.i.d or not;[7] Second, the case in which neither condition i) nor condition ii) hold. With reference to Judd and Guu (2001) we denote these two cases as Zero Jacobian and Singular Jacobian respectively.

3.1 Zero Jacobian case

In this case the elements of the Jacobian matrix of the dynamic system associated to the portfolio elements are exactly zero.[8]

In order to characterize the constant component of the portfolio (the zero-order portfolio) we need to evaluate equation (4) to second order. From Jin and Judd (2002) and Judd (1998) we know that in order to evaluate this second-order moment condition we need to evaluate each variable up to first order. To this order of approximation, the portfolio elements enter the model only through equation (2) as constants.[9]

[6] See any of our applications.

[7] I.e. condition i) holds but condition ii) might not hold.

[8] In the example in the next sectionthis will amount to block-partitioning the Jacobian matrix into a zero sub-matrix and a full-rank matrix.

[9] Notice furthermore that premium implied by the moment condition up to second order is constant, so up to this order there is no incentive for the agents to alter the portfolio shares.

Furthermore, if we want to solve for the first order portfolio dynamics, we need to evaluate equation (4) at least to third order. To evaluate the moment conditions up to third order suffices to evaluate its determinant variables up to second order. To this order of approximation, the portfolio elements are time varying.[10]

In the following section we make clear how this description of the solution translates in the singular-perturbation approach used by Judd and Guu (2001).

A conceptually simple solution technique, hence, consists of replacing the redundant moment condition with the linear equation

$$\alpha_{i,t} = \alpha_{i,0} + A z_t \tag{5}$$

and then of searching the unknown coefficient α_0 and elements of the vector A of the portfolio that satisfy the moment condition:[11]

$$\{\alpha_0, A\} = argmin \left\{ \left[\left(E_t U'(C^j_{t+1}) r_{x,t+1} \right) \Big|_{III-order} \right]^2 \right\}$$

where $X|_{III-order}$ denotes the third-order Taylor expansion of X. In the particularly simple case of the Zero Jacobian we can recursively break this problem into i) seraching for α_0 using the second-order accurate moment condition and ii) searching for the elements of A using the third-order accurate moment condition.[12]

As an illustration, here we describe the simple case of the zero-order portfolio.

The second order approximation of equation(s) (3), which (in log-deviation terms) yields

$$E_t \frac{U''(C^j_0)}{U'(C^j_0)} C^j_0 \widehat{C}^j_{t+1} \widehat{r}_{xt+1} + \widehat{r^s_{xt+1}} + \frac{1}{2} \widehat{r}^{\,2}_{xt+1} + \mathcal{O}\left(\|\varepsilon^3\|\right) = 0 \tag{6}$$

where the the term $\widehat{r^s_{xt+1}}$ would need to be evaluated using a second-order approximation to its policy function.

Taking the difference of each of the conditions that have been excluded from the solution of the DSGE model with respect to any of the corresponding conditions included in the solution (e.g. the one for country z) yields [13]

$$E_t \left(\frac{U''(C^j_0)}{U'(C^j_0)} C^j_0 \widehat{C}^j_{t+1} - \frac{U''(C^z_0)}{U'(C^z_0)} C^z_0 \widehat{C}^z_{t+1} - \widehat{Q}^{jz}_{t+1} \right) \widehat{r}_{xt+1} = 0 \tag{7}$$

[10] Notice also that to this order of approximation the premium is also time varying.

[11] This is in essence what is suggested by Van Wincoop and Tille (2007). They do not consider the Singular Jacobian case.

[12] Obviously, if one is only interested in the zero-order portfolio only i) should be carried out.

[13] The objective is to get rid of variables that require second order solutions, i.e. the linear term in $\widehat{r_{xt+1}}$.

where $\widehat{Q}^{jz}_{t+1}$ is the real exchange rate. This will allow us to obtain a system of $n-1 \times k-1$ equations that we can solve for the $n-1 \times k-1$ asset shares using simply a first-order approximation to the policy functions.

For the sake of simplicity define a new variable

$$dC_t = \left(\frac{U''(C_0^i)}{U'(C_0^i)} C_0^i \widehat{C}_t^i - \frac{U''(C_0^z)}{U'(C_0^z)} C_0^z \widehat{C}_t^z - \widehat{Q}^{jz}_{t+1} \right). \tag{8}$$

Condition (7) can then be simply re-written as

$$E_t dC_{t+1} \widehat{r}_{x,t+1} = 0 \tag{9}$$

To this purpose we need to extract from the state-space representation the equations relative to the variables dC_t and $\widehat{r}_{x,t+1}$. Recall that the state-space solution to first order can be written as

$$\begin{aligned} s_t &= F_1 x_{t-1} + F_2 s_{t-1} \\ y_t &= P_1 x_t + P_2 s_t \\ x_t &= N\, x_{t-1} + \varepsilon_t \end{aligned} \tag{10}$$

where, for the sake of simplicity we represent all variables (exogenous, states and controls) in the vector y_t. Then we have that

$$y_t = P_1 N x_{t-1} + P_2 s_t + P_1 \varepsilon_t \tag{11}$$

and

$$y_{t+1} = \Theta_1 x_{t-1} + \Theta_2 s_t + \Theta_3 \varepsilon_t + P_1 \varepsilon_{t+1} \tag{12}$$

$$\Theta = (P_1 N + P_2 F_1)\, N \tag{13}$$

$$\Theta_2 = P_2 F_2 \tag{14}$$

$$\Theta_3 = (P_2 F_1 + P_1) \tag{15}$$

Denote the position of the variables dC_t in the vector y_t by j_{dC} and the position of the variable $\widehat{r_{xt}}$ by j_{rx}. We know that arbitrage conditions imply that[14]

$$y_t(j_{rx}) = P_{1(j_{rx},:)} \varepsilon_t \tag{16}$$

Then we can write condition (9) as

$$E_t y(j_{dc}) y(j_{rx})' = E_t \left(\Theta_{1(dC,:)} x_{t-1} + \Theta_{2(dC,:)} s_t + \Theta_{3(dC,:)} \varepsilon_t + P_{1(dC,:)} \varepsilon_{t+1} \right) \varepsilon'_{t+1} P'_{1(j_{rx},:)} = 0 \tag{17}$$

which, after simplifying reduces to

$$E_t y(j_{dc}) y(j_{rx})' = P_{1(dC,:)} \Sigma P'_{1(j_{rx},:)} = 0 \tag{18}$$

[14]Following Matlab syntax, a ":" denotes all the elements along that particular dimension. E.g. $M_{(i,:)}$ denotes the row i of matrix M.

We notice that the matrix P_1will be a function of the steady state portfolio shares, as condition (18), in general, would be satisfied only by the optimal portfolio. Therefore, to solve for the optimal portfolio we need simply to solve condition (18). While in particularly simple cases this could be done analytically, in general we would need numerical methods to solve this equation.[15]

We apply this technique to the simple bond-economy model of Devereux and Sutherland (2008) for which we can also compute the solution using their method.[16] Figure 9 shows the residual of equation (18) as a function of the portfolio share α_0. The optimal portfolio share found by Devereux and Sutherland (2008) is at $\alpha_0 = -2.11864$.

3.2 Singular Jacobian

If to first order of approximation the dynamic component of the portfolio does not vanish (i.e. α_t is not multiplied by zero in the steady state), the zero-order portfolio and the first-order portfolio must be solved jointly. Further below we will show that it is still convenient to break the moment condition into two parts: i) the second-order accurate moment condition (Ω_2) and ii) the third-order accurate moment condition, conditional on the second-order condition being satisfied (Ω_3). In general we must use numerical methods to solve the fixed-point problem:

$$\begin{cases} \alpha_0 = \alpha(A) \\ A = A(\alpha_0) \\ \Omega_2^2 + \Omega_3^2 = 0 \end{cases}$$

A way to proceed is:

1. for initial guess $A^{(0)}$ use first-order solution of the model to search for $\alpha_0^{(1)}$ until $\Omega_2\left(A^{(0)}, \alpha^{(1)}\right) = 0$
2. use second-order solution of the model to search for $A^{(1)}$ until $\Omega_3\left(A^{(1)}, \alpha^{(1)}\right) = 0$
3. Continue until $\alpha_0^{(n+1)} = \alpha^{(n)}$

Notice that Ω_2 amounts to condition (20) while Ω_3 amounts to the term in square brackets in condition (31), derived further below.

[15]With reference to the simple model studied in the next section and in Devereux and Sutherland (2008), we would find that

$$P_{1(dC,:)} = (1-\beta)\left[\begin{array}{cc} 2\alpha + \frac{\beta}{1-\beta\zeta_y}, & 2\alpha + \frac{\beta}{1-\beta\zeta_{y^*}} \end{array}\right]$$

and

$$P_{1(j_{rx},:)} = \left[\begin{array}{cc} 1, & -1 \end{array}\right]$$

so that under $\Sigma = \left[\begin{array}{cc} \sigma_y^2 & 0 \\ 0 & \sigma_y^2 \end{array}\right]$ and $\zeta_y = \zeta_{y^*}$ we would have $\alpha = -\dfrac{\beta}{2(1-\beta\zeta_y)}$.

[16]Recall that this model is a special case of our three-country model displayed in the Appendix.

4 Comparing the Devereux and Sutherland and Judd and Guu solution methods

The solution methods used so far for the steady-state portfolio allocation are constructed without explicit reliance on the Implicit Function and Bifurcation Theorems"(henceforth IFT and BF respectively) Devereux and Sutherland (2008). On the contrary, Judd and Guu (2001) the foundation of the solution proposed by Judd and Guu (2001).[17] In this section we show that we can follow the same approach used in Judd and Guu (2001) to the simple dynamic model studied in Devereux and Sutherland (2008). In this way we are able to highlight the strong link between the two solution techniques. In doing so we extend the solution in Judd and Guu (2001) for the static asset market equilibrium to a dynamic framework. In the process, we spell out the mathematical conditions stressed by these outhors that are needed to ensure that the above system of equations yields a well defined solution.

Judd and Guu (2001) argue that the IFT cannot be used to approximate asset market equilibria because in the absence of uncertainty all assets are perfect substitutes, implying a continuum of equilibrium portfolio allocations. Hence, they resort to bifurcation methods to compute asset allocations for small deviations from the deterministic economy ("small risks").

We start by considering the simple economy in Devereux and Sutherland in which there are only two agents and two nominal bonds, for which the first order conditions characterizing the portfolio choice are given by:

$$
\begin{aligned}
0 &= E_t\left[(C_{t+1})^{-\rho}\, r_{x,t+1}\right] \\
0 &= E_t\left[\left(Y_{t+1} + Y^*_{t+1} - C_{t+1}\right)^{-\rho} r_{x,t+1}\right],
\end{aligned}
$$

where therefore $r_{x,t+1}$ is a scalar and we have used the economy resource constraint to substitute out consumption of the foreign agent, C^*. The other equi-

[17] For example, "Our solution approach relies on first-order and second-order approximations of the model, rather than the Implicit Function and Bifurcation Theorems, but the underlying theory described by Judd and Guu (2001) is applicable to our equilibrium solution" (Devereux and Sutherland, 2008, page 3)

librium conditions are

$$
\begin{aligned}
0 &= C_t^{-\rho} Z_t^* - \beta E_t \left\{ C_{t+1}^{-\rho} \frac{Y_{t+1}^*}{M_{t+1}^*} \right\} \\
0 &= (Y_t + Y_t^* - C_t)^{-\rho} Z_t^* - \beta E_t \left\{ \left(Y_{t+1} + Y_{t+1}^* - C_{t+1} \right)^{-\rho} \frac{Y_{t+1}^*}{M_{t+1}^*} \right\} \\
A_{t+1} &= (A_t + Y_t - C_t) \frac{Y_{t+1}^*}{M_{t+1}^* Z_t^*} + \alpha_t r_{x,t+1} \\
r_{x,t} &= \frac{\frac{Z_{t-1}^*}{Z_{t-1}} \frac{Y_t}{M_t} - \frac{Y_t^*}{M_t^*}}{Z_{t-1}^*} \\
\ln Y_{t+1} &= y_{t+1} = \zeta_y y_t + \sigma \varepsilon_{Y,t+1} \\
\ln Y_{t+1}^* &= y_{t+1}^* = \zeta_y y_t^* + \sigma \varepsilon_{Y^*,t+1},
\end{aligned}
$$

where Z_t is the nominal price of the bond and where we have substituted out the price level in terms of the quantity equation, and defined (total financial) wealth for the domestic agent A_t as in Lucas (1982) — note incidentally that this is an analytically more convenient expression than the one used by Devereux and Sutherland in terms of net savings:

$$
\begin{aligned}
W_t &= W_{t-1} \frac{Y_t^*}{Z_{t-1}^*} + \alpha_{t-1} r_{x,t} + Y_t - C_t, \\
A_t &= (W_t + Y_t - C_t).
\end{aligned}
$$

Following Judd (1998) and Schmitt-Grohé and Uribe (2004), the decision rules solving the above equilibrium conditions, can be generally expressed as functions of exogenous (y_t, y_t^*) and endogenous states (A_t), and the perturbation parameter σ :

$$
\begin{aligned}
C_t &= C\left(A_t, y_t, y_t^*; \sigma\right) \\
Z_t^* &= Z\left(A_t, y_t, y_t^*; \sigma\right) \\
r_{x,t} &= R\left(A_t, y_t, y_t^*; \sigma\right) \\
\alpha_t &= \alpha\left(A_t, y_t, y_t^*; \sigma\right) \\
A_{t+1} &= \Omega\left(A_t, y_t, y_t^*; \sigma\right),
\end{aligned}
$$

The four first order conditions thus define a functional equation $F\left(C\left(\cdot\right), Z\left(\cdot\right), \alpha\left(\cdot\right), R\left(\cdot\right), \sigma\right) = 0$. However, since they hold for any value of α in the nonstochastic steady state $(\sigma = 0)$, we cannot directly apply the implicit function theorem (IFT) to characterize the decision rules as it is customary in the perturbation approach — e.g. Schmitt-Grohé and Uribe (2004) and Lombardo and Sutherland (2007). To see this, differentiate both the portfolio and non portfolio equations in $F\left(\cdot\right)$ with respect to σ:

$$0 = \left[\begin{array}{c} -\rho C^{-1} \left(\begin{array}{c} C_\sigma + C_A E_t \left(\begin{array}{c} \alpha_\sigma \cdot 0 + \alpha \left(R_\sigma + R_A \Omega_A + R_y \varepsilon_{Y,t+1} + R_{y^*} \varepsilon_{Y^*,t+1} \right) + \\ \left(A + Y - C \right) \frac{\beta \varepsilon_{Y^*,t+1} - Z_\sigma}{\beta^2} - \frac{C_\sigma}{\beta} \end{array} \right) \\ + C_y E_t \varepsilon_{Y,t+1} + C_{y^*} E_t \varepsilon_{Y^*,t+1} \end{array} \right) \cdot 0 \\ + R_\sigma + R_A E_t \left(\begin{array}{c} \alpha_\sigma \cdot 0 + \alpha \left(R_\sigma + R_A \Omega_A + R_y \varepsilon_{Y,t+1} + R_{y^*} \varepsilon_{Y^*,t+1} \right) + \\ \left(A + Y - C \right) \frac{\beta \varepsilon_{Y^*,t+1} - Z_\sigma}{\beta^2} - \frac{C_\sigma}{\beta} \end{array} \right) \\ + R_y E_t \varepsilon_{Y,t+1} + R_{y^*} E_t \varepsilon_{Y^*,t+1} \end{array} \right]$$

$$0 = \left[\begin{array}{c} \rho C^{*-1} \left(\begin{array}{c} C_\sigma + C_A E_t \left(\begin{array}{c} \alpha_\sigma \cdot 0 + \alpha \left(R_\sigma + R_A \Omega_A + R_y \varepsilon_{Y,t+1} + R_{y^*} \varepsilon_{Y^*,t+1} \right) + \\ \left(A + Y - C \right) \frac{\beta \varepsilon_{Y^*,t+1} - Z_\sigma}{\beta^2} - \frac{C_\sigma}{\beta} \end{array} \right) \\ + \left(C_y - 1 \right) E_t \varepsilon_{Y,t+1} + \left(C_{y^*} - 1 \right) E_t \varepsilon_{Y^*,t+1} \end{array} \right) \cdot 0 \\ + R_\sigma + R_A E_t \left(\begin{array}{c} \alpha_\sigma \cdot 0 + \alpha \left(R_\sigma + R_A \Omega_A + R_y \varepsilon_{Y,t+1} + R_{y^*} \varepsilon_{Y^*,t+1} \right) + \\ \left(A + Y - C \right) \frac{\beta \varepsilon_{Y^*,t+1} - Z_\sigma}{\beta^2} - \frac{C_\sigma}{\beta} \end{array} \right) \\ + R_y E_t \varepsilon_{Y,t+1} + R_{y^*} E_t \varepsilon_{Y^*,t+1} \end{array} \right],$$

and

$$\begin{aligned} 0 &= -\beta \rho C^{-1} C_\sigma + Z_\sigma + \\ &+ \beta \left[\rho C^{-1} \left(\begin{array}{c} C_\sigma + C_y E_t \varepsilon_{Y,t+1} + C_{y^*} E_t \varepsilon_{Y^*,t+1} + \\ C_A \left(\begin{array}{c} \alpha_\sigma \cdot 0 + \alpha \left(R_\sigma + R_A \Omega_A + R_\varepsilon E_t \varepsilon_{Y,t+1} + R_{\varepsilon^*} E_t \varepsilon_{Y^*,t+1} \right) \\ + \left(A + Y - C \right) \frac{\beta E_t \varepsilon_{Y^*,t+1} - Z_\sigma}{\beta^2} - \frac{C_\sigma}{\beta} \end{array} \right) \end{array} \right) + E_t \varepsilon_{Y^*,t+1} \right] \\ 0 &= \beta \rho C^{*-1} C_\sigma + Z_\sigma + \\ &- \beta \left[\rho C^{*-1} \left(\begin{array}{c} C_\sigma + \left(C_y - 1 \right) E_t \varepsilon_{Y,t+1} + \left(C_{y^*} - 1 \right) E_t \varepsilon_{Y^*,t+1} + \\ C_A \left(\begin{array}{c} \alpha_\sigma \cdot 0 + \alpha \left(R_\sigma + R_A \Omega_A + R_\varepsilon E_t \varepsilon_{Y,t+1} + R_{\varepsilon^*} E_t \varepsilon_{Y^*,t+1} \right) \\ + \left(A + Y - C \right) \frac{\beta E_t \varepsilon_{Y^*,t+1} - Z_\sigma}{\beta^2} - \frac{C_\sigma}{\beta} \end{array} \right) \end{array} \right) + E_t \varepsilon_{Y^*,t+1} \right]. \end{aligned}$$

Provided C_A, C_y, C_{y^*} are well defined, and given that

$$r_{x,t} = \frac{\frac{Z_{t-1}^*}{Z_{t-1}} \frac{Y_t}{M_t} - \frac{Y_t^*}{M_t^*}}{Z_{t-1}^*}$$

implies $R_A = 0$ and $R_y = -R_{y^*} = \beta^{-1}$, the last two equations at $\sigma = 0$ further simplify:

$$\begin{aligned} 0 &= Z_\sigma + \beta \rho C^{-1} C_A \left(\alpha_\sigma \cdot 0 + \alpha R_\sigma - \left(A + Y - C \right) \frac{Z_\sigma}{\beta^2} - \frac{C_\sigma}{\beta} \right) \\ 0 &= Z_\sigma - \beta \rho C^{*-1} C_A \left(\alpha_\sigma \cdot 0 + \alpha R_\sigma - \left(A + Y - C \right) \frac{Z_\sigma}{\beta^2} - \frac{C_\sigma}{\beta} \right). \end{aligned}$$

Clearly these two equations imply that $Z_\sigma = 0$, and $\alpha \beta R_\sigma = C_\sigma$, provided α_σ is well-defined.

To verify the assumption on α_σ, and compute R_σ and C_σ we need to consider the two portfolio Euler equations. Provided again C_A, C_y, C_{y^*} are well defined, when evaluated at $\sigma = 0$ they simplify to:

$$0 = \alpha_\sigma \cdot 0 + R_\sigma + \frac{E_t\left(\varepsilon_{Y,t+1} - \varepsilon_{Y^*,t+1}\right)}{\beta}$$
$$0 = \alpha_\sigma \cdot 0 + R_\sigma + \frac{E_t\left(\varepsilon_{Y,t+1} - \varepsilon_{Y^*,t+1}\right)}{\beta},$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \alpha_\sigma + \begin{bmatrix} 1 \\ 1 \end{bmatrix} R_\sigma.$$

If α_σ is to be well defined, given that it is multiplied by 0, the last two terms have to be equal to 0 as well (see Judd and Guu, 2001). Intuitively, the only way the derivative α_σ can be well defined is if it is the solution of the indeterminate form $\frac{0}{0}$, which can be dealt with using l'Hospital's rule. Then it must be the case that $R_\sigma = 0$, and thus it is also $C_\sigma = 0$.

We still wish to try to determine the steady state portfolio allocation α as a function of uncertainty σ. The condition above ensuring the existence of α_σ defines a candidate bifurcation point at $\sigma = 0$, as the portfolio allocation from determinate becomes indeterminate — the number of solutions for α for the first order conditions changes as σ increases from 0. The broad idea of bifurcation analysis is to provide conditions to find a point where the branch of interesting (e.g. unique) solutions to a system of equations crosses the "trivial" branch of (indeterminate) solutions, at which point the nontrivial solution can be characterized as an implicit function of an underlying parameter (e.g. σ), for which a Taylor series approximation can be found. In order to do so, following Judd and Guu, we substitute a second order expansion of $r_{x,t+1}$ in σ in the two portfolio equations:

$$0 = E_t\left\{C_{t+1}^{-\rho}\left[\left(\beta^{-1}\left(\varepsilon_{Y,t+1} - \varepsilon_{Y^*,t+1}\right) + R_\sigma\right)\sigma + \frac{1}{2}\mathcal{R}\left(\sigma\right)\sigma^2\right]\right\}$$
$$0 = E_t\left\{\left(Y_{t+1} + Y_{t+1}^* - C_{t+1}\right)^{-\rho}\left[\left(\beta^{-1}\left(\varepsilon_{Y,t+1} - \varepsilon_{Y^*,t+1}\right) + R_\sigma\right)\sigma + \frac{1}{2}\mathcal{R}\left(\sigma\right)\sigma^2\right]\right\};$$

where $\mathcal{R}(\sigma)$ represents the risk premium of the domestic bond relative to the foreign bond and is thus a function of σ. This substitution — assuming a quadratic gauge function in the perturbation parameter σ, see Judd (1998), Ch. 15 — ensures that the first two conditions of Theorem 4 in Judd and Guu for the existence of a bifurcation point in $\mathbb{R}^n$ are satisfied, namely that:

$$H_z\left(z_0, \sigma = 0\right) = 0_{2x2}$$
$$H_\sigma\left(z_0, \sigma = 0\right) = 0_{2x1},$$

where $z = \left[\mathcal{R}(\sigma), \alpha(\cdot, \sigma)\right]$ and the (analytic) function $H(\cdot)$ is defined by the two portfolio first order conditions. This theorem ensures the existence of two implicit functions $\mathcal{R}(\sigma) \neq 0$ and $\alpha(\cdot, \sigma) \neq 0$ for $\sigma \neq 0$, such that $\lim_{\sigma \to 0} \alpha(\cdot, \sigma) = \alpha(0)$

is well defined. Furthermore, these functions are analytic and can be approximated by a Taylor series. In order, to see this, dividing for, and differentiating with respect to σ, now yields

$$H_\alpha \alpha_\sigma + H_{\mathcal{R}} \mathcal{R}_\sigma + H_\sigma = 0 \tag{19}$$

that is,

$$0 = -\rho\beta^{-1}C^{-1}E_t \left[\left(\begin{array}{c} C_A \alpha_\sigma \cdot 0 + \beta^{-1} C_A \left(\begin{array}{c} \alpha \left(\varepsilon_{Y,t+1} - \varepsilon_{Y^*,t+1} \right) + \\ \left(A + Y - C \right) \varepsilon_{Y^*,t+1} \end{array} \right) \\ + C_y \varepsilon_{Y,t+1} + C_{y^*} \varepsilon_{Y^*,t+1} \end{array} \right) \left(\varepsilon_{Y,t+1} - \varepsilon_{Y^*,t+1} \right) \right]$$
$$+ \frac{1}{2} \left(\sigma \mathcal{R}_\sigma + \mathcal{R} \right)$$
$$0 = \rho\beta^{-1}C^{*-1}E_t \left[\left(\begin{array}{c} C_A \alpha_\sigma \cdot 0 + \beta^{-1} C_A \left(\begin{array}{c} \alpha \left(\varepsilon_{Y,t+1} - \varepsilon_{Y^*,t+1} \right) + \\ \left(A + Y - C \right) \varepsilon_{Y^*,t+1} \end{array} \right) \\ + \left(C_y - 1 \right) \varepsilon_{Y,t+1} + \left(C_{y^*} - 1 \right) \varepsilon_{Y^*,t+1} \end{array} \right) \left(\varepsilon_{Y,t+1} - \varepsilon_{Y^*,t+1} \right) \right]$$
$$+ \frac{1}{2} \left(\sigma \mathcal{R}_\sigma + \mathcal{R} \right),$$

where

$$H_\alpha \left(\alpha, \mathcal{R}, \sigma = 0 \right) = 0 \cdot C_A \left[\begin{array}{c} -\rho\beta^{-1}C^{-1} \\ \rho\beta^{-1}C^{*-1} \end{array} \right]$$
$$H_{\mathcal{R}} \left(\alpha, \mathcal{R}, \sigma = 0 \right) = \frac{1}{2} \left[\begin{array}{c} 0 \\ 0 \end{array} \right],$$

and

$$H_\sigma \left(\alpha, \mathcal{R}, \sigma = 0 \right) = 0_{2x1}$$
$$= \left[\begin{array}{c} -\rho\beta^{-1}C^{-1}E_t \left[\left(\begin{array}{c} \beta^{-1} C_A \left(\begin{array}{c} \alpha \left(\varepsilon_{Y,t+1} - \varepsilon_{Y^*,t+1} \right) + \\ \left(A + Y - C \right) \varepsilon_{Y^*,t+1} \end{array} \right) \\ + C_y \varepsilon_{Y,t+1} + C_{y^*} \varepsilon_{Y^*,t+1} \end{array} \right) \left(\varepsilon_{Y,t+1} - \varepsilon_{Y^*,t+1} \right) \right] \\ \rho\beta^{-1}C^{*-1}E_t \left[\left(\begin{array}{c} C_A \alpha_\sigma \cdot 0 + \beta^{-1} C_A \left(\begin{array}{c} \alpha \left(\varepsilon_{Y,t+1} - \varepsilon_{Y^*,t+1} \right) + \\ \left(A + Y - C \right) \varepsilon_{Y^*,t+1} \end{array} \right) \\ + \left(C_y - 1 \right) \varepsilon_{Y,t+1} + \left(C_{y^*} - 1 \right) \varepsilon_{Y^*,t+1} \end{array} \right) \left(\varepsilon_{Y,t+1} - \varepsilon_{Y^*,t+1} \right) \right] \end{array} \right] + \tfrac{1}{2} \mathcal{R}$$

defines the bifurcation point in the unknowns $\mathcal{R}(0)$ and $\alpha(0)$. It is worthwhile to notice that this expression effectively amounts to taking a second order Taylor approximation of the portfolio equations, as required by the general solution approach discussed in the previous section, and also by Devereux and Sutherland.

Taking the difference of the two equations to get rid of $\mathcal{R}$ we have:

$$0 = E_t \left[C^{-1} \left(\begin{array}{c} C_A \left(\begin{array}{c} \alpha \left(\varepsilon_{Y,t+1} - \varepsilon_{Y^*,t+1} \right) + \\ \left(A + Y - C \right) \varepsilon_{Y^*,t+1} \end{array} \right) \\ + \beta C_y \varepsilon_{Y,t+1} + \beta C_{y^*} \varepsilon_{Y^*,t+1} \end{array} \right) \left(\varepsilon_{Y,t+1} - \varepsilon_{Y^*,t+1} \right) \right] +$$
$$E_t \left[C^{*-1} \left(\begin{array}{c} C_A \left(\begin{array}{c} \alpha \left(\varepsilon_{Y,t+1} - \varepsilon_{Y^*,t+1} \right) + \\ \left(A + Y - C \right) \varepsilon_{Y^*,t+1} \end{array} \right) \\ + \beta \left(C_y - 1 \right) \varepsilon_{Y,t+1} + \beta \left(C_{y^*} - 1 \right) \varepsilon_{Y^*,t+1} \end{array} \right) \left(\varepsilon_{Y,t+1} - \varepsilon_{Y^*,t+1} \right) \right].$$

Under the same assumptions as in Devereux-Sutherland of a symmetric steady state ($C = C^*$) and uncorrelated shocks the above expression simplifies to:

$$0 = 2C_A\alpha \left(\sigma^2_{Y,t+1} + \sigma^2_{Y^*,t+1}\right) + \beta \left((2C_y - 1)\, \sigma^2_{Y,t+1} - (2C_{y^*} - 1)\, \sigma^2_{Y^*,t+1} \right),$$

which allows to express steady state α as a function of the first order approximation of the consumption decision rule:

$$\alpha = -\beta \frac{(2C_y - 1)\, \sigma^2_{Y,t+1} - (2C_{y^*} - 1)\, \sigma^2_{Y^*,t+1}}{2C_A \left(\sigma^2_{Y,t+1} + \sigma^2_{Y^*,t+1}\right).}.$$

Solving for α thus requires knowledge of the policy functions for consumption and excess returns up to first order. The last step is to verify that the Jacobian $H_{\sigma(\alpha,\mathcal{R})}\left(\alpha(\sigma), \mathcal{R}(\sigma), \sigma = 0\right)$ is nonsingular, as required by Theorem 4 in Judd and Guu. Here it is easy to verify that this condition is met by differentiating $H_\sigma(\alpha, \mathcal{R}, \sigma)$ above to obtain:

$$Det\left(H_{\sigma(\alpha,\mathcal{R})}\left(\alpha(0), \mathcal{R}(0), 0\right)\right) = \begin{vmatrix} -\rho \dfrac{C_A}{\beta^2 C}\left(\sigma^2_{Y,t+1} + \sigma^2_{Y^*,t+1}\right) & \frac{1}{2} \\ \rho \dfrac{C_A}{\beta^2 C}\left(\sigma^2_{Y,t+1} + \sigma^2_{Y^*,t+1}\right) & \frac{1}{2} \end{vmatrix} \neq 0$$

if $C_A \neq 0$.

Finally, to show that the solution is exactly the same as in Devereux-Sutherland we need to derive C_A, C_y, C_{y^*}. This can be done easily by differentiating the system $F\left(C(\cdot), Z(\cdot), \alpha(\cdot), R(\cdot), \sigma\right)$ with respect to A_t, y_t and y_t^*, to obtain:

$$\begin{aligned} C_A &= 1 - \beta \\ \frac{1 - \beta\zeta_y}{\beta} C_y &= \frac{1 - \zeta_y}{2} + \frac{1 - \beta}{\beta} \\ \frac{1 - \beta\zeta_y}{\beta} C_{y^*} &= \frac{1 - \zeta_y}{2}. \end{aligned}$$

The following two observations are in order. First, while the portfolio allocation does not appear in the above solutions, this will not generally be the case and the first order approximation will depend on the steady state value of α. Second, the (unknown) first order approximation terms of the portfolio allocation rule $(\alpha_A, \alpha_y, \alpha_{y^*})$ all appear multiplied by 0, as in the case of α_σ above. Thus, they will also be well defined if, loosely speaking, they are expressed as a solution of the indeterminate form $\frac{0}{0}$. In turn, it can be shown that this condition will be always satisfied by the solutions of the unknown derivatives of the other decision rules $\left(C(\cdot), Z(\cdot), R(\cdot)\right)$, obtained by standard perturbations methods based on differentiation of $F\left(C(\cdot), Z(\cdot), \alpha(\cdot), R(\cdot), \sigma\right)$.

We are now in a position to solve for α; for instance, assuming $\sigma^2_{Y,t+1} = \sigma^2_{Y^*,t+1}$ as in Devereux and Sutherland (2008), the counterpart of the expression in their Section 4.1.2 obtains:

$$\frac{\alpha}{\beta} = -\frac{C_y - C_{y^*}}{2(1-\beta)} = -\frac{1}{2(1-\beta\zeta_y)}.$$

To summarize, in this section we have shown that the method by Judd and Guu, based on a rigorous application of bifurcation techniques, can be extended to dynamic economies, allowing to determine the steady state portfolio allocation as a function of the first order terms of consumption decision rules and excess returns. For a specific dynamic economy also studied by Devereux and Sutherland (2008) we have formally shown the coincidence of the solutions under the two approaches. This is due to the fact that the key condition used to solve for the portfolio allocation is exactly the same moment condition, obtained from a second order Taylor approximation of the portfolio equations by Devereux and Sutherland, and from the use of a quadratic gauge in the same set of equations by Judd and Guu. The use of a second order Taylor expansion or a quadratic gauge in a perturbation approach amount to the same thing. In both cases the Bifurcation Theorem allows us to say whether a solution exists at a second order of approximation.

5 Steady-state portfolios

In this section we show how the near-stochastic steady-state optimal portfolio solution of Devereux and Sutherland (2008) can be generalized to the case of more than two-agents (and more than two assets).[18]

Under some general conditions, Devereux and Sutherland show that since expected excess returns are equal to zero up to first order, the term $\sum_{i=1}^{k-1} \alpha_{i,t-1}^{j} (r_{i,t} - r_{B,t})$ will only be a function of the unexpected shocks in the approximate solution around the steady state — in the case of our model economy, the vector of innovations of exogenous processes ε. Moreover, they show that in the case of 2 countries the steady state optimal portfolio will be implicitly defined by the following moment conditions obtained by taking a second order approximation of the portfolio first order conditions around a non-stochastic steady state:

$$E_{t-1}\left[(C_{1,t} - C_{2,t})\, r_{x,i,t}\right] = 0,$$

where $r_{x,i,t} = r_{i,t} - r_{B,t}$, $i = 1...k-1$. Under the assumption of homoschedastic shocks, the above conditions will be the same for any period. Devereux and Sutherland (2008) show that the term $\sum_{i=1}^{k-1} \alpha_{i,t-1}^{j} (r_{i,t} - r_{B,t})$ can be replaced with the auxiliary i.i.d. variable ξ_t, so that a solution for the approximated equilibrium around the non stochastic steady state will yield policy rules for the vector of

[18]From now on variables will denote log-deviations from their steady-state value, except for net wealth (W) and individual assets (α) which are measured relative to output.

excess returns $r_{x,t}$ and for $\Delta_t = (C_{1,t} - C_{2,t})$ which will be functions of ξ_t and innovations ε_t.[19]

Since $\xi_t \approx \alpha' r_{x,t}$, where α denotes the steady-state value of α_t, the auxiliary variable could be substituted out yielding expressions in terms of fundamentals innovations for $\Delta_t = \widetilde{D}\varepsilon_t$ and $r_{x,t} = \widetilde{R}\varepsilon_t$. Formally,[20]

$$\begin{aligned} \widehat{R}_{x,t} &= R_1\xi_t + R_2\varepsilon_t \\ \Delta_t &= d_1\xi_t + D_2\varepsilon_t, \end{aligned}$$

so that substituting out ξ_t and re-arranging yields

$$\begin{aligned} \widetilde{R} &= (I - R_1\alpha')^{-1} R_2 \\ \widetilde{D} &= d_1\alpha' (I - R_1\alpha')^{-1} R_2 + D_2 \end{aligned}$$

Then the above time-invariant portfolio moment conditions will amount to the following matrix equation:

$$\underbrace{0}_{k\times 1} = \underbrace{\widetilde{R}\Sigma\widetilde{D}'}_{(k\times 1)(e\times e)(e\times 1)} \tag{20}$$

implicitly defining the steady-state unknown elements of the vector α, representing the gross holdings of foreign assets and liabilities for country 1, excluding the reference asset. The position of the latter will be derived from the assumed level of steady state net foreign assets — we will assume throughout that this is zero for all countries.

Equation (20) can be easily solved for α, i.e.[21]

$$\alpha = (R_2\Sigma D_2' R_1' - d_1 R_2 \Sigma R_2')^{-1} (R_2\Sigma D_2').$$

In the case of more than two agents, to take into account the effects of asset returns on the wealth distribution across agents, we will have to keep track of the holdings of n-1 agents, and include the relevant moment conditions from their portfolio optimization problems. For each n-1 couple of agents it is more convenient to write the moment conditions as

$$\begin{aligned} \underbrace{0}_{1\times k} &= \underbrace{\widetilde{D^i}'\Sigma\widetilde{R}'}_{(1\times e)(e\times e)(e\times k)} \\ 0 &= \left[\underbrace{D_1^i}_{1\times(n-1)}\underbrace{\alpha'}_{(n-1)\times k}\underbrace{(I - R_1\overline{\alpha}')^{-1}}_{k\times k}\underbrace{R_2}_{k\times e} + \underbrace{D_2^i}_{1\times e}\right]' \Sigma R_2' \left((I - R_1\alpha')^{-1}\right)', \end{aligned}$$

[19] The marginal utility differential Δ will also be a function of state variables, like the wealth distribution.

[20] Δ_t would depend also on state variables. This term, though, would drop out in the cross product with $r_{x,t}$.

[21] In deriving this expression we have made use of the fact that $\left((I - R_1\overline{\alpha}')^{-1}\right)' \overline{\alpha} \equiv \overline{\alpha}\left((I - \overline{\alpha}' R_1)^{-1}\right)'$.

where as indicated above, $\overline{\alpha}$ is now a $k \times (n-1)$ matrix. Rearranging the above expression yields:

$$\underbrace{\left(D_2^i \mathcal{R} R_1 - D_1^i\right)}_{1\times(n-1)} \underbrace{\overline{\alpha}'}_{(n-1)\times k} \quad = \quad \underbrace{D_2^i}_{1\times e} \underbrace{\mathcal{R}}_{e\times k},$$

where $\mathcal{R} \equiv \Sigma R_2' \left(R_2 \Sigma R_2'\right)^{-1}$.

Collecting all the conditions yields the following system:

$$\underbrace{\begin{bmatrix} \left(D_2^1 \mathcal{R} R_1 - D_1^1\right) \\ \vdots \\ \left(D_2^i \mathcal{R} R_1 - D_1^i\right) \\ \vdots \\ \left(D_2^{n-1} \mathcal{R} R_1 - D_1^{n-1}\right) \end{bmatrix}}_{(n-1)\times(n-1)} \overline{\alpha}' = \underbrace{\begin{bmatrix} D_2^1 \\ \vdots \\ D_2^i \\ \vdots \\ D_2^{n-1} \end{bmatrix}}_{(n-1)\times e} \underbrace{\mathcal{R}}_{exk},$$

or

$$\underbrace{\Omega}_{(n-1)\times(n-1)} \quad \alpha' = \underbrace{\widetilde{D}_2'}_{(n-1)\times e} \underbrace{\mathcal{R}}_{e\times k},$$

where $\Omega \equiv \widetilde{D}_2 \left(I \otimes \mathcal{R} R_1\right) - \widetilde{D}_1$, $\widetilde{D}_2 \equiv \begin{bmatrix} D_2^{1\prime} & \ldots & D_2^{i\prime} & \ldots & D_2^{n-1\prime} \end{bmatrix}$ and $\widetilde{D}_1' \equiv \begin{bmatrix} D_1^{1\prime} & \ldots & D_1^{i\prime} & \ldots & D_1^{n-1\prime} \end{bmatrix}$.

A solution for the steady state portfolio is given by

$$\alpha' = \Omega^{-1} \widetilde{D}_2' \mathcal{R}.$$

Notice that with two agents the above formula yields

$$\alpha' = -\frac{\left(D_2 \Sigma R_2'\right)\left(R_2 \Sigma R_2'\right)^{-1}}{\left(d_1 - \left(D_2 \Sigma R_2'\right)\left(R_2 \Sigma R_2'\right)^{-1} R_1\right)},$$

which can be shown to be equal to the expression derived by Devereux and Sutherland (2008).

6 Portfolio dynamics

We are now in the position to solve for the portfolio dynamics. We first present the case of two agents and one asset to re-cast the Devereux and Sutherland (2007a) result into our notation. The multiple-agent multiple-asset case will be a straightforward extension.

6.1 The two-agents one-asset case

Devereux and Sutherland (2007a) show that in order to obtain a second-order accurate solution for the portfolio dynamics, one has to take a third order approximation of the consumption Euler equation of each agent.[22] In particular, with reference to their equation (26) we have

$$E_t\left[-\rho\left(C_{t+1}-C^*_{t+1}\right)r_{x,t+1}+\frac{\rho^2}{2}\left(C^2_{t+1}-C^{*,2}_{t+1}\right)r_{x,t+1}-\frac{\rho}{2}\left(C_{t+1}-C^*_{t+1}\right)\left(r^2_{1,t+1}-r^2_{2,t+1}\right)\right]=0 \tag{21}$$

In order to evaluate this expression we need the solution for consumption and the return on the assets at most to the second order of accuracy.

As shown in Lombardo and Sutherland (2007) the second order solution for the consumption differential between the two agents can be written as

$$\begin{aligned}(C-C^*) &= \overbrace{D_1\xi+D_2\varepsilon+D_3z}^{\text{First order part}} \\ &\quad +\underbrace{D_0+D_4vec\left(\varepsilon\varepsilon'\right)+z'\hat{D}_5\varepsilon+D_6vec\left(zz'\right)}_{\text{Second order part}}\end{aligned} \tag{22}$$

and

$$\begin{aligned}r_x &= \overbrace{R_1\xi+R_2\varepsilon}^{\text{first order part}} \\ &\quad +\underbrace{E\left[r_x\right]-R_4\overrightarrow{\Sigma}+R_4vec\left(\varepsilon\varepsilon'\right)+z'\hat{R}_5\varepsilon+R_6vec\left(zz'\right)}_{\text{Second order part}}\end{aligned} \tag{23}$$

where all the variables have the same timing, $\overrightarrow{\Sigma}=E\left[vec\left(\varepsilon\varepsilon'\right)\right]$, and where $z'=\left[x_t,\, s_{t+1}\right]$; x_t being the $e\times 1$ vector of shocks and s_t the $l\times 1$vector of endogenous state variables.[23]

Devereux and Sutherland (2007a), show that the portfolio dynamics can be described as a linear transformation of the state variables, i.e.

$$\alpha_{t-1}=\gamma'\,z_t \tag{24}$$

where γ in this case is a $(l+e)\times 1$ vector of coefficients to be determined.

The second order approximation of the budget constraint of each agent (equation (2)) generates a term in the cross product of the portfolio and the

[22] This subsection does not add anything to Devereux and Sutherland (2007a) except reproducing their calculations using matrix notation in place of tensor notation.

[23] See the Appendix for a second-order state-space representation of the solution. The matrices in equations (22) and (23) correspond to transformations of the rows of the P matrices given in the Appendix. Notice that, up to second order, $R_0=E\left[r_x\right]-R_3z-R_4\overrightarrow{\Sigma}-R_6vec\left(zz'\right)$ and that, since up to first order r_x is *i.i.d.*, $R_3z=0$. Notice also that it is possible to re-write terms like $R_5vec\left(\varepsilon z'\right)$ as $z'\hat{R}_5\varepsilon$. See the Appendix for further details. The arrows denote vectorization.

excess returns: $\alpha_{t-1}r_{x,t}$. Following Devereux and Sutherland (2007a), we can express this product as an i.i.d. variable

$$\xi_{t+1} \equiv \alpha_t r_{x,t+1} = \gamma z_{t+1} r_{x,t+1} \tag{25}$$

Since this relation involves only first order variables, we can use the first order part of equation (23) in the latter expression to get (dropping time subscripts)

$$\xi = \gamma' z R_2 \varepsilon = R_2 \varepsilon z' \gamma \tag{26}$$

Replace this into (22) and (23)

$$\begin{aligned} (C - C^*) \quad &= \quad D_0 + D_1 z' \gamma R_2 \varepsilon + D_2 \varepsilon + D_3 z \\ &\quad + D_4 \overrightarrow{\varepsilon\varepsilon} + z' \hat{D}_5 \varepsilon + D_6 \overrightarrow{zz} \end{aligned} \tag{27}$$

and

$$\begin{aligned} r_x \quad &= \quad E\left[r_x\right] - R_4 \overrightarrow{\Sigma} + R_1 z' \gamma R_2 \varepsilon + R_2 \varepsilon \\ &\quad + R_4 \overrightarrow{\varepsilon\varepsilon} + z' \hat{R}_5^i \varepsilon. \end{aligned} \tag{28}$$

As this expression involves first order terms in C and $r_{1,2}$we need the first order solution to these variables, i.e.

$$\begin{array}{ll} C = C_2^H \varepsilon + C_3^H z & C^* = C_2^F \varepsilon + C_3^F z \\ r_1 = R_2^H \varepsilon + R_3^H z & r_2 = R_2^F \varepsilon + R_3^F z \end{array}$$

where by the $i.i.d.$nature of r_x must be that $R_3^F = R_3^H$.

Taking cross-products of these equations we have, e.g.

$$C^2 = \left(C_2^H \otimes C_2^H\right) \overrightarrow{\varepsilon\varepsilon} + \left(C_3^H \otimes C_3^H\right) \overrightarrow{zz} + \left[\left(C_2^H \otimes C_3^H\right) + \left(C_3^H \otimes C_2^H\right) P_v\right] \overrightarrow{\varepsilon z} \tag{29}$$

where P_v is a vector-permutation matrix. For convenience we re-write these cross products as

$$\begin{array}{ll} C^2 = C_{2\otimes2}^H \overrightarrow{\varepsilon\varepsilon} + C_{3\otimes3}^H \overrightarrow{zz} + C_{2\otimes3}^H \overrightarrow{\varepsilon z} & C^{*,2} = C_{2\otimes2}^F \overrightarrow{\varepsilon\varepsilon} + C_{3\otimes3}^F \overrightarrow{zz} + C_{2\otimes3}^F \overrightarrow{\varepsilon z} \\ r_1^2 = R_{2\otimes2}^H \overrightarrow{\varepsilon\varepsilon} + R_{3\otimes3}^H \overrightarrow{zz} + R_{2\otimes3}^H \overrightarrow{\varepsilon z} & r_2^2 = R_{2\otimes2}^F \overrightarrow{\varepsilon\varepsilon} + R_{3\otimes3}^F \overrightarrow{zz} + R_{2\otimes3}^F \overrightarrow{\varepsilon z} \end{array}$$

and

$$C^2 - C^{*,2} = C_{2\otimes2}^{H-F} \overrightarrow{\varepsilon\varepsilon} + C_{3\otimes3}^{H-F} \overrightarrow{zz} + C_{2\otimes3}^{H-F} \overrightarrow{\varepsilon z}$$

and

$$r_1^2 - r_2^{*,2} = R_{2\otimes2}^{H-F} \overrightarrow{\varepsilon\varepsilon} + R_{2\otimes3}^{H-F} \overrightarrow{\varepsilon z}$$

where we have defined e.g. $C_{2\otimes2}^{H-F} \equiv \left(C_{2\otimes2}^H - C_{2\otimes2}^F\right)$.

Consider one addendum of equation (21) at a time (and abstract from the ρ coefficient for the time being), i.e.

$$D_0 r_x + D_1 z' \gamma R_2 \varepsilon r_x + D_2 \varepsilon r_x + D_3 z r_x + D_4 \overrightarrow{\varepsilon\varepsilon} r_x + z' \hat{D}_5 \varepsilon r_x + D_6 \overrightarrow{zz} r_x \tag{30}$$

$$\left(C_{2\otimes 2}^{H-F}\overrightarrow{\varepsilon\varepsilon}+C_{3\otimes 3}^{H-F}\overrightarrow{zz}+C_{2\otimes 3}^{H-F}\overrightarrow{\varepsilon z}\right)\quad\times\quad\varepsilon' R_2' \tag{31}$$

$$\left(D_2\varepsilon+D_3 z\right)\quad\times\quad\left(R_{2\otimes 2}^{H-F}\overrightarrow{\varepsilon\varepsilon}+R_{2\otimes 3}^{H-F}\overrightarrow{\varepsilon z}\right)' \tag{32}$$

Let's start with equation (30). Recall that to first order $r_x = R_2\varepsilon$, while, to a first order $C - C^* = C_2^{H-F}\varepsilon + C_3^{H-F}z$

Notice that since $D_2\varepsilon\varepsilon' R_2' = 0$, and assuming that third moments of the shocks are zero, the (conditional on time t) expected value of equation (30) reduces to (omitting the expectation operator)

$$\left[R_2\varepsilon\varepsilon' R_2' D_1 z'\gamma + z'\hat{D}_5\varepsilon\varepsilon' R_2'\right]+\left(z'\hat{R}_5\varepsilon\varepsilon' D_2'\right)+$$
$$+D_2\varepsilon E\left[r_x\right]+\left(E\left[r_x\right]-\left(\overrightarrow{\Sigma}\right)' R_4'+\left(\overrightarrow{\varepsilon\varepsilon}\right)' R_4'\right)\left(D_3 z\right)$$

Notice that

$$\left(E\left[r_x\right]-\left(\overrightarrow{\Sigma}\right)' R_4'+\left(\overrightarrow{\varepsilon\varepsilon}\right)' R_4'\right)\left(D_3 z\right)\quad=\left(E\left[r_x\right]\right)\left(D_3 z\right)$$

Furthermore notice that the sum of the second order expansion of the Euler equations we can derive an expression for $E\left[r_x\right]$. So for, say the home country, have

$$C^{-\rho}\beta\left(\widehat{r}_1-\widehat{r}_2+\frac{1}{2}\widehat{r}_1^2-\frac{1}{2}\widehat{r}_2^2\right)-\rho\beta C^{-\rho}\left(\widehat{C}\right)\left(\widehat{r}_1-\widehat{r}_2\right)=0.$$

Adding this to the foreign counterpart yields,

$$E\left[r_x\right]=\frac{\rho}{2}E\left[\left(C_2^H+C_2^F\right)\varepsilon\varepsilon' R_2'+\left(C_3^H+C_3^F\right)z\varepsilon' R_2'\right]-\frac{1}{2}\left(R_{2\otimes 2}^{H-F}\overrightarrow{\varepsilon\varepsilon}+R_{2\otimes 3}^{H-F}\overrightarrow{\varepsilon z}\right)$$

implying that

$$\left(E\left[r_x\right]\right)\left(D_3 z\right)=\rho\left(C_2^H\right)\left[D_3 z\Sigma R_2'\right]-\frac{1}{2}R_{2\otimes 2}^{H-F}\overrightarrow{\varepsilon\varepsilon}D_3 z$$

and that

$$D_2\varepsilon E\left[r_x\right]=\rho\left(C_3^H+C_3^F\right)zR_2\varepsilon\varepsilon' D_2'-\frac{1}{2}D_2\varepsilon\left(R_{2\otimes 3}^{H-F}\overrightarrow{\varepsilon z}\right)=-\frac{1}{2}D_2\varepsilon\left(R_{2\otimes 3}^{H-F}\overrightarrow{\varepsilon z}\right)=0$$

where we have used the fact that

$$D_2\varepsilon\left(R_{2\otimes 3}^{H-F}\overrightarrow{\varepsilon z}\right)=D_2\varepsilon\left(R_2^H\varepsilon\varepsilon' R_2^H-R_2^F\varepsilon\varepsilon' R_2^F\right)=D_2\varepsilon R_2^H\varepsilon\varepsilon' R_2'=0$$

and that that $D_3 z$ is a scalar and that $\left(C_2^H+C_2^F\right)\Sigma R_2' = 2C_2^H\Sigma R_2' + D_2\Sigma R_2'$.

the first addendum of equation (21) gives[24]

$$E_t \left[R_2 \varepsilon\varepsilon' R_2' D_1 z' \gamma + z' \hat{D}_5 \varepsilon\varepsilon' R_2' \right] +$$
$$\left(z' \hat{R}_5 \varepsilon\varepsilon' D_2' \right) + \rho \left(C_2^H \right) \left[D_3 z \Sigma R_2' \right] - \frac{1}{2} R_{2\otimes 2}^{H-F} \overrightarrow{\varepsilon\varepsilon} D_3 z$$

As for the second addendum one can show that it reduces to

$$\begin{aligned} R_2 \varepsilon\varepsilon' \left(z' \otimes I \right)' \left(C_{2\otimes 3}^{H-F} \right)' &= 2 R_2 \varepsilon \left[\left(C_2^H \varepsilon z' C_3^{H\prime} \right) - \left(C_2^F \varepsilon z' C_3^{F\prime} \right) \right] \\ &= 2 C_2^H D_3 z \Sigma R_2' \end{aligned} \tag{33}$$

And for the third have

$$\left(R_{2\otimes 2}^{H-F} \overrightarrow{\varepsilon\varepsilon} \right) \left(D_3 z \right) + D_2 \varepsilon\varepsilon' \left(z' \otimes I \right)' \left(R_{2\otimes 3}^{H-F} \right)'$$

where

$$\begin{aligned} D_2 \varepsilon\varepsilon' \left(z' \otimes I \right)' \left(R_{2\otimes 3}^{H-F} \right)' &= 2 \quad D_2 \varepsilon \left[R_2^H \varepsilon z' R_3^{H\prime} - R_2^F \varepsilon z' R_3^{F\prime} \right] \\ &= 2 D_2 \varepsilon\varepsilon' R_2^{H\prime} z' R_3^{H\prime} - D_2 \varepsilon\varepsilon' R_2^H z' R_3^{F\prime} \\ &= 2 D_2 \varepsilon\varepsilon' R_2^{H\prime} z' \left[R_3' \right] = 0 \end{aligned}$$

and

$$\left(R_{2\otimes 2}^{H-F} \overrightarrow{\varepsilon\varepsilon} \right) \left(D_3 z \right) = \left[R_2^H \varepsilon\varepsilon' R_2^{H\prime} - R_2^F \varepsilon\varepsilon' R_2^{F\prime} \right] \left(D_3 z \right)$$

Taking all terms together yields

$$\begin{aligned} -E_t \left[R_2 \varepsilon\varepsilon' R_2' D_1 z' \gamma + z' \hat{D}_5 \varepsilon\varepsilon' R_2' \right] &+ \\ - \left(z' \hat{R}_5 \varepsilon\varepsilon' D_2' \right) - \rho \left(C_2^H \right) \left[D_3 z \Sigma R_2' \right] + \frac{1}{2} R_{2\otimes 2}^{H-F} \overrightarrow{\varepsilon\varepsilon} D_3 z &\quad + \\ \rho \left[C_2^H D_3 z \Sigma R_2' \right] &\quad + \\ -\frac{1}{2} \left[\left(R_{2\otimes 2}^{H-F} \right) \overrightarrow{\varepsilon\varepsilon} \left(D_3 z \right) \right] & \end{aligned}$$

Simplifying gives

$$E_t z' \left\{ \left[\gamma R_2 \Sigma R_2' D_1 + \hat{D}_5 \Sigma R_2' \right] + \left(\hat{R}_5 \Sigma D_2' \right) \right\} = 0$$

This must be valid for all possible z i.e.

$$\left\{ \left[\gamma R_2 \Sigma R_2' D_1 + \hat{D}_5 \Sigma R_2' \right] + \left(\hat{R}_5 \Sigma D_2' \right) \right\} = 0$$

or

$$\gamma = -\frac{\hat{D}_5 \Sigma R_2' + \hat{R}_5 \Sigma D_2'}{R_2 \Sigma R_2' D_1}$$

which is the formula derived in Devereux and Sutherland (2007a).

[24] Notice that $D_2 \varepsilon\varepsilon' R_2' = 0$ implies that $D_2 \varepsilon\varepsilon' R_2^{H\prime} = D_2 \varepsilon\varepsilon' R_2^{F\prime}$ and that $C_2^H \varepsilon\varepsilon' R_2' = C_2^F \varepsilon\varepsilon' R_2'$.

6.2 Two-agents multiple-assets case

In order to extend the result of Devereux and Sutherland (2007a) to the multiple-asset case we notice that α_t and $r_{x,t}$ are now $k-1$ vectors.

Now we have $k-1$ different Euler equations, one for asset

$$\begin{aligned} (C - C^*) &= D_0 + D_1\xi + D_2\varepsilon + D_3 z \\ &\quad + D_4 vec\left(\varepsilon\varepsilon'\right) + z'\hat{D}_5\varepsilon + D_6 vec\left(zz'\right) \end{aligned} \tag{34}$$

and, for the $i-th$ asset

$$\begin{aligned} r_{x,i} &= E\left[r_x\right] - R_4^i\overrightarrow{\Sigma} + R_1^i\xi + R_2^i\varepsilon \\ &\quad + R_4^i vec\left(\varepsilon\varepsilon'\right) + z'\hat{R}_5^i\varepsilon + R_6^i vec\left(zz'\right) \end{aligned} \tag{35}$$

The first order solution to the vector of excess returns can be written as

$$r_x = \underbrace{R_2}_{(k-1)\times e} \varepsilon \tag{36}$$

where $R_2' = \left[R_2^{1,\prime} \dots R_2^{k-1,\prime}\right]$. Then the cross-product of assets and their excess return can still be written as

$$\xi_{1\times 1} = z'\gamma R_2\varepsilon \tag{37}$$

where γ is a $l \times (k-1)$ matrix of coefficients.

Then, by analogy with the results shown earlier we have

$$E_t z'\left\{\left[\gamma R_2\varepsilon\varepsilon' R_2^{i,\prime} D_1 + \hat{D}_5\varepsilon\varepsilon' R_2^{i,\prime}\right] + \left(\hat{R}_5^i\varepsilon\varepsilon' D_2'\right)\right\} = 0$$

or using the fact that

$$\left(\hat{R}_5^i\varepsilon\varepsilon' D_2'\right) = vec\left(I\,\hat{R}_5^i\varepsilon\varepsilon' D_2'\right) = \left(D_2\varepsilon\varepsilon' \otimes I_{l\times l}\right) vec\left(\hat{R}_5^i\right) \tag{38}$$

$$\underbrace{E_t z'\left\{\left[\gamma R_2\varepsilon\varepsilon' R_2^{i,\prime} D_1 + \hat{D}_5\varepsilon\varepsilon' R_2^{i,\prime}\right] + \left(D_2\varepsilon\varepsilon' \otimes I_{l\times l}\right) vec\left(\hat{R}_5^i\right)\right\}}_{l\times 1} = 0_{s\times 1}$$

This must be valid for all possible z i.e.

$$\underbrace{\left\{\left[\gamma R_2\Sigma R_2^{i,\prime} D_1 + \hat{D}_5\Sigma R_2^{i,\prime}\right] + \left(D_2\Sigma \otimes I_{l\times l}\right) vec\left(\hat{R}_5^i\right)\right\}}_{l\times 1} = 0_{s\times 1}$$

$$\vdots$$

$$\underbrace{\left\{\left[\gamma R_2\Sigma R_2^{k-1,\prime} D_1 + \hat{D}_5\Sigma R_2^{k-1,\prime}\right] + \left(D_2\Sigma \otimes I_{l\times l}\right) vec\left(\hat{R}_5^{k-1}\right)\right\}}_{l\times 1} = 0_{s\times 1}$$

stack column wise all the $k-1$ conditions and get

$$\underbrace{\left\{\left[\gamma R_2 \Sigma R_2' D_1 + \hat{D}_5 \Sigma R_2'\right] + \left(D_2 \Sigma \otimes I_{l\times l}\right) R'_{5,l\cdot e\times(k-1)}\right\}}_{l\times k-1} = 0_{s\times k} \tag{39}$$

or

$$\gamma = -\left[\hat{D}_5 \Sigma R_2' + \left(D_2 \Sigma \otimes I_{l\times l}\right) R'_{5,l\cdot e\times(k-1)}\right] \left(R_2 \Sigma R_2' D_1\right)^{-1}$$

6.3 The multiple-agents multiple-assets case

Notice that now we have $n-1$ i.i.d. terms

$$\xi^j_{1\times 1} = \alpha_j' r_x : \ j = 1 \ldots n-1$$

where α_j and r_x are $(k-1)\times 1$.

Furthermore, now we have $k-1$ different equations for the assets excess return and $n-1$ for the consumption differentials.

Our solution strategy is to find the optimal portfolio condition agent-by-agent and then stack them together to solve the simultaneous portfolio problem. The counterpart of the optimal portfolio condition as found in the previous section (equation (39)) is the following for agent i

$$E_t \underbrace{\left[D_1^{i,1}\gamma_1 R_2 \varepsilon\varepsilon' R_2 + \cdots + D_1^{i,n-1}\gamma_{n-1} R_2 \varepsilon\varepsilon' R_2 + \hat{D}_5^i \varepsilon\varepsilon' R_2' + \left(D_2^i \Sigma \otimes I_{l\times l}\right) R_5'\right]}_{l\times(k-1)} = 0$$

where $D_1^{i,j}$ is a scalar, γ_j is a $l\times(k-1)$ matrix $(j = 1\ldots(n-1))$.

More compactly we can write

$$\mathcal{D}_1^i \begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_2 \end{bmatrix} = -\left[\mathcal{D}_5^i \Sigma R_2' + \mathcal{D}_2^i R'_{5,l\cdot e\times(k-1)}\right] \left(R_2 \Sigma R_2'\right)^{-1}$$

where $\mathcal{D}_1^i \equiv \left[\begin{array}{ll} D_1^{i,1} I_{l\times l} & \ldots D_1^{i,2} I_{l\times l} \end{array}\right]$, $\mathcal{D}_2^i \equiv \left(D_2^i \Sigma \otimes I_{l\times l}\right)$ and $\mathcal{D}_5^i = \hat{D}_5^i$.

We can stack these equations (for agent $i = 1\ldots(n-1)$) row by row to get

$$\mathcal{D}_1 \begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_2 \end{bmatrix} = -\left[\mathcal{D}_5 \Sigma R_2' + \mathcal{D}_2 R'_{5,l\cdot e\times(k-1)}\right] \left(R_2 \Sigma R_2'\right)^{-1}$$

where $\mathcal{D}_1$is $(n-1)\cdot l \times (n-1)\cdot l$. Then have

$$\begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_2 \end{bmatrix} = -\mathcal{D}_1^{-1}\left[\mathcal{D}_5 \Sigma R_2' + \mathcal{D}_2 R'_{5,l\cdot e\times(k-1)}\right] \left(R_2 \Sigma R_2'\right)^{-1} \tag{40}$$

7 Application 1: Monetary shocks and optimal portfolio choice in a three-country model with nominal bonds

In this section we apply our formulae for the solution of the optimal portfolio to a three-country version of the endowment-economy model used by Devereux and Sutherland (2008, 2007a).

The model consists of three symmetric economies populated by identical agents having preferences and constraints as described by equations (1) and (2). These agents can only trade in three nominal bonds. Money and endowments, in each country, follow an exogenous AR(1) process. Prices are determined by a simple equation of exchange i.e.[25]

$$P_{i,t} Y_{i,t} = M_{i,t} \tag{41}$$

where $i = \{a, b, c\}$ denotes the country. Table 1 reports the non-linear model.

The variance of the stochastic innovations is denoted by $\sigma_{Y,i}$ and $\sigma_{M,i}$, $i = \{a, b, c\}$.

7.1 Steady-state portfolio

This model remains sufficiently simple to allow for an analytical representation of the solution for the steady-state portfolio. For the sake of comparison, the steady-state portfolio for the two-country two-bonds version of our model is given by

$$\alpha = -\frac{1}{2}\frac{1}{(1-\beta\zeta_Y)}\left(\frac{\sigma_{a,Y}+\sigma_{b,Y}}{\sigma_{a,Y}+\sigma_{b,Y}+\sigma_{a,M}+\sigma_{b,M}}\right) \tag{42}$$

For the three-country case the solution will be a 2×2 matrix. The rows of α correspond to country a and c respectively and the columns correspond to the bonds issued in the currency of the country a and c respectively. Country b is the reference country and its position can be inferred in relation to the net-wealth of country a and c.

Extending the results of Devereux and Sutherland (2008) we can show that in the case of our three-country three-bond model we have[26]

[25] Here velocity is set to 1. Alternatively we could have assumed a constant money supply and a stochastic velocity (money demand shocks). The results would be the same.

[26] One can show that

$$R_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad R_2 = \begin{bmatrix} -1 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 \end{bmatrix}$$

$$D_1 = \rho(1-\beta)\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad D_2 = \rho\frac{1-\beta}{1-\beta\zeta_Y}\begin{bmatrix} 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$

Notice that in this model the risk-aversion parameter ρ cancels out in the formula for the portfolio.

$$\alpha^D = 3\left[(\sigma_{M,a}+\sigma_{Y,a})\left((\sigma_{M,b}+\sigma_{Y,b})+(\sigma_{M,c}+\sigma_{Y,c})\right)+(\sigma_{M,b}+\sigma_{Y,b})(\sigma_{M,c}+\sigma_{Y,c})\right](1-\zeta_Y\beta) \tag{43}$$

$$\alpha^N_{1,1} = -2(\sigma_{M,b}+\sigma_{Y,c}+\sigma_{M,c}+\sigma_{Y,b})\sigma_{Y,a}+(\sigma_{M,c}+2\sigma_{Y,c})\sigma_{Y,b}+\sigma_{M,b}\sigma_{Y,c} \tag{44}$$

$$\alpha^N_{1,2} = \left((2\sigma_{M,b}+\sigma_{Y,c}+\sigma_{Y,b})\sigma_{Y,a}+\sigma_{M,a}(\sigma_{Y,c}-\sigma_{Y,b})+(\sigma_{Y,b}+\sigma_{M,b})\sigma_{Y,c}\right) \tag{45}$$

$$\alpha^N_{2,2} = -\left(\sigma_{M,b}+2(\sigma_{Y,c}+\sigma_{Y,b})\right)\sigma_{Y,a}+(\sigma_{M,a}+2\sigma_{Y,c})\sigma_{Y,b}+2(\sigma_{M,a}+\sigma_{M,b})\sigma_{Y,c} \tag{46}$$

$$\alpha^N_{2,1} = \left((\sigma_{Y,a}+2\sigma_{M,b}+\sigma_{Y,b})\sigma_{Y,c}+\sigma_{M,c}(\sigma_{Y,a}-\sigma_{Y,b})+(\sigma_{Y,b}+\sigma_{M,b})\sigma_{Y,a}\right) \tag{47}$$

and

$$\alpha_{i,j} = \frac{\alpha^N_{i,j}}{\alpha^D} \tag{48}$$

In the special case of identical variances across countries we have

$$\alpha = \frac{\sigma_Y}{3(\sigma_M+\sigma_Y)(1-\zeta_Y\beta)}\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \tag{49}$$

This is to say that, each country splits its portfolio in a short position in domestic assets and a long position (of equal amount) in foreign assets. The foreign position is then equally divided among the foreign-currency bonds.

In the special case of the reference country (country b) having zero variance of endowment and policy, we obtain

$$\alpha = \frac{1}{3}\begin{bmatrix} -2\dfrac{\sigma_{Y,a}}{(\sigma_{M,a}+\sigma_{Y,a})(1-\zeta_Y\beta)} & \dfrac{\sigma_{Y,c}}{(\sigma_{M,c}+\sigma_{Y,c})(1-\zeta_Y\beta)} \\ \dfrac{\sigma_{Y,a}}{(\sigma_{M,a}+\sigma_{Y,a})(1-\zeta_Y\beta)} & -2\dfrac{\sigma_{Y,c}}{(\sigma_{M,c}+\sigma_{Y,c})(1-\zeta_Y\beta)} \end{bmatrix} \tag{50}$$

The portfolio that we obtain for generic variances highlights an important difference between the two-country two-asset model and the three-country three-asset version. In particular, the general case highlight the interdependence between the different types of risk. In the two-country version (Cf. equation (42)) the risk associated with monetary policy affects the total portfolio holding: in the limit, as either of the monetary policy variances goes to infinity, the gross positions are run to zero.

This is not the case for the three-country model. For example, one can show that

$$\lim_{\sigma_{M,1}\to\infty}\alpha_{i,j} = \begin{cases} 0 & \text{if } j=1 \\ \dfrac{1}{3}\dfrac{\sigma_{Y,c}-\sigma_{Y,b}}{(1-\zeta_Y\beta)(\sigma_{Y,b}+\sigma_{Y,c}+\sigma_{M,b}+\sigma_{M,c})} & \text{if } i=1 \\ -\dfrac{1}{3}\dfrac{2\sigma_{Y,c}+\sigma_{Y,b}}{(1-\zeta_Y\beta)(\sigma_{Y,b}+\sigma_{Y,c}+\sigma_{M,b}+\sigma_{M,c})} & \text{otherwise} \end{cases} \tag{51}$$

so that all countries would still hold gross positions in the bonds issued in the currencies of the countries with finite variances. Notice though, that there is an asymmetry between these holdings: The holding $\alpha_{1,2}$ (i.e. country a's holding of bonds in country c's currency) would be zero if the other two countries (b and c) experience the same degree of endowment volatility. In other words, as the monetary policy of a country becomes too volatile, this country would hold nominal foreign assets only to the extent that they can hedge the foreign relative risk: going short (long) in country c (b) currency if the its endowment is less (more) volatile than country b's (c's). Country c on the contrary would have non-zero positions in country c's and country b's currency (going short in the first and long in the second).[27]

More in general, these results show that in this simple model no country goes long in its own currency. The position in the foreign asset will instead depend on the relative volatility of the foreign countries and on the domestic volatility of monetary policy.

Finally, if monetary policy is perfectly correlated across countries, the portfolio reduces to

$$\alpha = \frac{1}{3\left(\zeta_Y \beta - 1\right)} \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \tag{52}$$

independently of the variance of the endowments. This result highlights the fact that it is not the absolute volatility of monetary policy that matters for portfolio allocation but rather its volatility relative to that of other countries' monetary policies.

7.2 Portfolio Dynamics

An analytical description of the dynamic properties of the optimal portfolio would be too involved to offer (new) useful insights.[28] In this section, therefore, we give numerical examples of the dynamic response of the portfolio to shocks.

As benchmark parameterization we use the following values:

Description	Symbol	Value
Discount factor	β	0.98
Risk aversion	ρ	1
Persistence of money shock	$\zeta_{M_{\{a,b,c\}}}$	0.6
Persistence of output shock	$\zeta_{Y_{\{a,b,c\}}}$	0.9
Variance of shocks	$\sigma_{Y,M_{\{a,b,c\}}}$	1

As discussed earlier, the dynamics of the portfolio, up to second order of accuracy, can be described as a linear mapping from exogenous and endogenous state variables.

[27] One can show that $\frac{\partial \alpha_{1,2}}{\partial \sigma_{M,a}} < 0$, $\frac{\partial \alpha_{2,1}}{\partial \sigma_{M,c}} < 0$, $\frac{\partial \alpha_{2,1}}{\partial \sigma_{M,a}} \lesseqqgtr 0$ depending on the relative size of the variances.

[28] Devereux and Sutherland (2007a) derive the analytical response coefficients of the optimal portfolio, up to the underlying variances and for a unit degree of risk aversion. Their solution is insightful as it separates the portfolio level-effect induced by net-wealth movements, from the hedging effect due to exogenous shocks.

As is apparent from the analysis of Devereux and Sutherland (2007a), changes in the portfolio composition should be interpreted as deriving from two main factors: The level effect and the hedging effect.[29] The latter reflects changes in the sensitivity, to shocks, of the cross-country consumption differential. A given change in expected returns must bring about an equal change in the marginal utility of the trading countries. As utility is concave, changes in marginal utility can only be brought about by large changes in consumption when the level of consumption is high. At high levels of consumption the covariance of the returns and relative consumption is larger (other things equal), thus increasing hedging motives.

The level effect is due to the fact that increases in net-wealth, other things equal, translate into increases in holdings of all bonds. For example, in our model a positive money supply shock in country a improves its net-wealth position. Therefore, the share of all currencies in country a's portfolio increases (in particular $\alpha_{t,(1,i)} - \alpha_{1,i} = 0.94$).[30] The opposite is true for the other countries, which experience a net-wealth deterioration.

Upon a domestic endowment shock in country a, the share of domestic currency in country a's portfolio decreases (by -2.0857) while the share of foreign currency bonds in country a's portfolio is reduced (by-0.0871 per foreign currency). After the first period the shares increase monotonically. The response of the portfolio derives from the combined effect of the desire to reinforce the hedging position (as relative consumption has increased) and a change in net-wealth.

Figure 1 shows the response to country a's endowment shock of country a's holding of country a's currency bond and of country c's currency bond (the response of home net-wealth is also shown). Figure 2 shows the response to the same shock of country c's portfolio.

7.2.1 Net-wealth persistence and portfolio adjustment

The permanent effect of the shocks, due to incomplete markets and to approximation of the model around the non-stochastic steady-state, seems to contradict the statement that there is a unique steady-state portfolio. Under the alternative assumption that there are only endowment shocks, we would reproduce complete markets, so that the optimal choice of the portfolio composition would make the model stationary.[31] A way to evaluate the sensitivity of the model to the unit root under the incomplete market assumption, would be to adopt the suggestion by Schmitt-Grohé and Uribe (2003) for small open economy models.[32] One of the solution that these authors consider is to impose that the agents discount factor is a function of deviations of (country-wide) consumption from the steady

[29] This decomposition is also emphasized by Van Wincoop and Tille (2007) and referred to as portfolio growth and portfolio reallocation, respectively.

[30] We return in the next section to the problem non-stationarity of the responses.

[31] This fact is used, for example, by Engel and Matsumoto (2008) to derive the optimal portfolio composition under complete-markets.

[32] Another way to eliminate the unit root would be to assume that agents die with a positive probability. This solution is adopted, for example, by Van Wincoop and Tille (2007).

state value. We specify the discount factor as $\beta_t = \beta(c_t - \bar{c}) = (1 + c_t - c)^{(-\psi)}$. Under this assumption, the size of the largest root lying within the unit circle is inversely proportional to ψ. [33]

In this model, the steady-state portfolio is unaffected by this modification.[34] The dynamics of the portfolio, instead, is affected by ψ. The response of the portfolio to any shock is smaller (in absolute terms) and shorter lived the larger is ψ. In the special case of i.i.d. shocks (and for monetary shocks in general), the only dynamics imparted to the portfolio is the endogenous dynamics deriving from the evolution of net wealth. In this case, for example, a positive endowment shock in country a induces a current account surplus and, therefore, an increase in the net-foreign-asset position of this country. So, despite a reduction in holding of own-currency bonds on impact (due to the temporary increase in sensitivity of consumption differentials), after the first period the increase in wealth implies an increase in holding of all types of bonds for as long as wealth is above the steady-state level. In the extreme case of a one-period-lived net-wealth increase, the portfolio would return to its steady state after one period.

7.3 Welfare

It is easy to use the solution for the optimal portfolio to compute countries' respective welfare. It is important to notice, first, that the dynamic behavior of the portfolio, up to second order, does not have consequences for aggregate welfare, when the latter is measured using the expected (conditional or not) life-time utility of the households. This is because the only place where the portfolio dynamics appears, is the budget constraint of the households. There it shows up lagged (i.e. the amount that is carried over from last period) and multiplied by excess returns. Since excess returns are i.i.d. up to first order, the conditional expectation of this product must be zero.[35]

On the contrary, the steady-state composition of the portfolio has important effects on welfare. The following Table reports the welfare gains (in steady-state consumption units) for each country in adopting the optimal portfolio relative to the case of one single bond (of the reference country b).[36]

[33] It should be noticed that our stochastic discount factor does not affect the degree of relative risk aversion. The discount factor is a function of aggregate consumption so that $-\frac{U''C}{U'} = \rho$. Nevertheless, the sensitivity of aggregate consumption differentials, in general and even at first order, is affected by the stochastic discount factor. Hence, the sensitivity of the portfolio to ψ is not simply due to the reduction is persistence of net wealth.

A better way to eliminate the unit root would be to evaluate the model around a point that takes into account the role of risk (see Ljungqvist and Sargent (2000, ch. 14)). This solution is not trivial and we leave it to future research.

[34] While we don't have a symbolic representation of this case, numerical experiments showed that the matrices $D_1$and D_2 that enter the formula for the steady-state portfolio depend ψ. As is happens for the parameter of risk aversion, though, ψ cancels out (see footnote 26).

[35] This implies that to appreciate welfare gains from portfolio dynamics we need to look at order of approximation higher than second for welfare.

[36] The infinite discounted sum of the conditional mean of the period utility is re-scaled by $(1 - \beta)$.

	Country a	Country b	Country c
Welfare gain	18.29%	35.24%	18.29%

The table shows that, as should be expected, all countries experience a welfare improvement when the steady-state portfolio is chosen optimally. It is also evident that the gain is larger for country b. This can be rationalized by considering that in the single-bond case, country a and country c run a positive net-wealth position on average (up to second order of accuracy). This means that they must have a long position in the foreign asset (country b's currency bond). Although quantitatively different, this is the position that they hold in the three-bond case. Country b, on the contrary does not have the possibility of going long in foreign currency.[37]

7.4 Some implications for exchange rate regimes

This example, although admittedly quite simplified, allows nevertheless to make an interesting point related to the literature on currency union, precisely the branch spurred by Neumeyer (1998). His paper shows, in a two-country model, that to the extent that having an independent currency expands the set of assets trades thus fostering international risk sharing, the costs of having a suboptimally volatile monetary policy will not be enough to establish a monetary union.

Neumeyer's argument emerges clearly from the solution for optimal portfolio allocation in the two country case. The cost from lack of diversification entailed by setting up a monetary union can be matched only by an independent monetary policy with an arbitrarily large volatility of shocks – in both cases agents will be forced to hold a zero equilibrium position in both currencies.

It is clear however that a world economy with several countries may substantially alter this result, as foregoing monetary independence will reduce only some of the diversification opportunities provided by nominal state uncontingent assets, rather than all of them, as in the case of two countries only.

Ravenna (2005) and Clerc et al. (2008) also show that imperfect monetary policy credibility or lack of commitment possibilities at the national level can generate positive gains from joining a currency union.[38] Also in these cases, the balance of pros and cons of pegging the currency could be tilted against the currency union if exchange rates can be used to hedge idiosyncratic risks. In a multi-country context, though, each country might need to have access only to a subset of independent currencies to find the currency union attractive. In a multi-country context, it might be that if a subset of currencies remains independent, the gain from joining the currency union might be positive. This

[37] The two-country version of this model gives the opposite result: i.e. the reference country (b) has a smaller gain from portfolio diversification. Indeed in this case, in expectations, country a takes a very large short position in the foreign currency, while the optimal portfolio prescribes a long position.

[38] See also Giavazzi and Pagano (1988).

raises strategic issues, since the optimal currency area would have fewer members than existing countries.

It would be straightforward to extend the model described above, for instance by including an endogenous cost of monetary policy volatility beyond that implied by the presence of nominal bonds, so as to provide a formal analysis of these trade-offs.

8 Application 2: A two-country model with equities and bonds

For the second application we use a two-country version of the model used in Dedola and Straub (2008) (see Table 2 in the Appendix).

This model is a simple extension of the previous model. Now there are only two countries. Each country's total income is composed of an endowment (as in the previous section) and of a dividend stream. Only claims on the dividend stream can be traded. The model allows for home-bias in consumption (denoted by μ) and partial substitutability of domestic and foreign goods (elasticity denoted by θ). In this model we introduce the stochastic discount factor discussed in the previous section.

Benchmark parameterization

Our benchmark parameterization displays the following values

Description	Symbol	Value
(Non-Stochastic) Discount Factor	β	0.98
Elasticity of Substitution	θ	3
Home Bias in Consumption	μ	0.8
Risk Aversion	ρ	2
Persistence of the Shocks	ζ	0.5
Elasticity of Stochastic Discount Factor	ψ	0

Furthermore, we assume that the share of dividends in total income is 15%.

8.1 Steady-State portfolio

There are two main points that we want to make in relation to the steady-state portfolio. The first is that, contrary to the previous application, the persistence of the net-wealth deviation from steady state has effects on the steady state portfolio. The second is that when we allow only for dividend and endowment shocks we have the result of perfect equity home bias only when this two shocks have identical persistence.

8.1.1 Net-Wealth persistence

Figure 3 shows that introducing dividend shocks and equities alongside bonds generates a steady-state effect of the stochastic discount factor. It is clear though that the sensitivity of the portfolio to ψ is relatively small so that even quantitatively the steady-state results are independent of the persistence of net-wealth.

8.1.2 Persistence of the shocks and equity home bias

Coeurdacier et al. (2007), in a two-period two-country endowment model, show that full home-equity bias can be produced, for example, when only output and redistributive shocks are present. By setting to zero the variance of the monetary shocks in our model, perfect home-bias in equities holds only in the special case of equal persistence of the dividend and endowment shocks (although they can have different variance). The persistence of the shocks, as discussed in the previous application, is important. For example, predictable changes in endowments, affect the sensitivity of consumption differential to shocks. Only when dividend and endowment shocks are equally persistent, the home equity provides a perfect hedge once bonds insure against exchange rate risk.[39]

Figure 6 shows the portfolio of country a for different relative values of the endowment and dividend shock. For equities the shares are shown: i.e. 1 means that 100% of the equities are held by country a. For this example, the persistence of the dividend shock is fixed at 0.5 and we vary the persistence of the endowment shock. We see that when the endowment shock is less persistent than the dividend shock, country a holds less than 100% of domestic equities and holds a positive share of foreign equities.[40] When endowment shocks are more persistent than dividend shocks, country a goes short in foreign equities and long in domestic equities.

8.2 Portfolio dynamics

Using equation (40) we can derive the response of bonds and equities to the exogenous shocks of the model. Figures (4) and (5) show the response of country a's portfolio to a domestic dividend shock and to a domestic endowment shock, respectively, in the "complete-market" case: i.e. when only these two sources of risk are present and the two shocks are equally persistent.

The steady-state portfolio of this particular case displays a short position in home currency bonds and full equity home bias.[41] After either of the two shocks has hit the economy net-wealth falls and so do all the gross positions. The persistence of these changes is entirely dictated by the persistence of the shocks. Furthermore the change in value of the equity positions is due to the

[39] See (Engel and Matsumoto, 2008) for a discussion on the role of bonds in insuring against the exchange rate risk.

[40] Short selling of equities is not ruled out.

[41] This corresponds to the point $\zeta_Y \cdot \zeta_D^{-1} = 1$ of Figure (6).

change in their price (as opposed to change in shares).[42]

These Figures also show an important results of the complete-market case: If the agents choose the optimal steady-state portfolio, the portfolio dynamics is mean-reverting.[43]

If we introduce monetary shocks along side the dividend and endowment shocks we reproduce the incomplete-market case. Now the optimal steady-state portfolio is contingent on the existence of a further source of risk. Its composition, therefore, differs from the composition of the optimal portfolio under complete-markets: e.g. country a goes short in domestic currency bonds and in foreign equities. It goes long on domestic equities.[44]

The portfolio dynamics in this case shows non-stationarity, as is evident from Figures (7) and (8). In this particular case, net-wealth improves after a dividend or endowment shock. In the short run holding of domestic assets increases while holding of foreign assets falls.

9 Application 3: The Ramsey optimal policy in a two-country sticky-price model with bonds

In this section we show an example of a model in which the portfolio shares enter the model also not multiplied by the excess return, and in particular multiplied by a variable that is not i.i.d.. This fact violates the necessary conditions for the application of the technique suggested by Devereux and Sutherland (2008) and by Van Wincoop and Tille (2007). We use here the sticky-price two-country model presented in Benigno and Benigno (2006), extended by introducing two bonds, one per currency and two shocks per country: a cost-push shock and a productivity shock. We don't give here all the details of the model as we only need to focus on few dimensions of it. [45]

[42] As the model is written in the Appendix, the share of country a's equities held by country a's households is $\frac{\alpha_2}{Z_a} - 1$ and their share of country b's equities is $\frac{\alpha_3}{Z_b}$.

[43] It is important to stress that if the steady-state portfolio is not optimal, either of these shocks would produce a permanent departure of net-wealth from the steady-state value. Having assumed equal persistence of the shocks is irrelevant for this result.

[44] In particular, the value of the portfolio is $\alpha = \begin{bmatrix} -0.2135 & 0.3870 & -0.3870 \end{bmatrix}$, where the elements of α correspond to domestic bonds, home equities and foreign equities, respectively.

[45] The reader should refer to Benigno and Benigno (2006) for details. In essence the model is a two-country production economy with labor as only factor of production, prices set à la Calvo (1983) and Dixit-Stiglitz consumption aggregators. The point we want to make here is to show that our technique can solve this type of models. This is true independently of the specific values assigned to the parameters of the model. Nevertheless, in the particular case studied here we assume a probability of not adjusting prices of 0.8; discount factor of 0.99 (quarterly frequency); intra-temporal elasticity of substitution (cross-country) of 1.5; mark-up of 1.11 subsidized with a tax (not crucial for the results); elasticity of intertemporal substitution of 1 and elasticity of labor supply of 1. We assume that government spending is zero and that the shocks have a coefficient of autocorrelation of 0.9. The size of the variance is identical for all shocks and the absolute value of these variances is not relevant for our results as condition (18) makes clear.

In particular, the Ramsey (cooperative) policy problem can be described as

$$\max_{Y_t,\alpha_t,r_{x,t},Y_t^*} E_t \sum_{i=0}^{\infty} \beta^i \left\{ nU\left(C_{t+i}, L_{t+i}\right) + (1-n)U\left(C_{t+i}^*, L_{t+i}^*\right) \right\} \tag{53}$$

subject to

$$E_t F\left(Y_t, Y_{t+1}, Y_{t-1}, Y_t^*, Y_{t+1}^*, Y_{t-1}^*, r_{x,t}, r_{x,t}\alpha_{t-1}, \varepsilon_t\right) = 0 \tag{54}$$

where β is the discount factor, $U(\cdot)$ are the utility functions of the representative agents of each country, n is the size of the first country (home) and $1-n$ is the size of the second country (foreign), C_t is consumption, L_t is labor (asterisks denoting foreign variables), ε_t is the vector of all the shocks while the vectors Y_t and Y_t^* contain all the variables of the model, excluding portfolio shares (α_t) and excess real returns ($r_{x,t}$). The function $F(\cdot)$ contains all the first-order conditions of the agents' optimization problem as well as the resources constraints. In particular, $F(\cdot)$ contains a subset of efficient portfolio conditions that is linearly independent (This is justified by the constrained qualification requirement in the Kuhn-Tucker Theorem (e.g. Sundaram (1996)).[46]

The Ramsey optimal policy is described by the set of FOCs for the problem (53)-(54). In particular, among these we have two conditions that relate specifically to the portfolio problem, namely

$$F_{r_{x,t}}(\cdot) = \cdots + \lambda_{BC,t}\alpha_{t-1} + \cdots = 0 \tag{55}$$

and

$$F_{\alpha_t}(\cdot) = E_t \beta \lambda_{BC,t+1} r_{x,t+1} = 0 \tag{56}$$

where $\lambda_{BC,t}$ is the Lagrange multiplier associated with the net-saving equation of the form (2). The second equation has exactly the form of the orthogonality condition (3).

It is possible to show that in general, in a symmetric (non-stochastic) steady state equilibrium$\lambda_{BC} = 0$.[47] In this case the dynamics of α is not determinate to first order of approximation. More in general though, this example shows that the dynamics of the portfolio would be determinate to first order whenever a policy maker sets its policy by solving an optimization problem subject to the decentralized-economy constraints and if $\lambda_{BC} \neq 0$. Here we only focus on the symmetric case. Even in this case, though, we cannot apply Devereux and Sutherland (2008) technique as $\lambda_{BC,t}$ is not an i.i.d. variable.

[46] Abstracting from the portfolio allocation (e.g. setting α to any constant value), after removing one of the portfolio efficiency conditions and after including a description of the monetary policy, this model could be solved with standard methods as done for example in Benigno and Benigno (2006).

[47] Consider the abstract policy problem of transferring dW units of wealth from the home agent to the foreign agent for consumption purposes, then the total differentiation of the Lagrange equation around the steady state, i.e. $0.5U'(C)\,dC + 0.5U(C^*)\,dC^* + \lambda_{BC}dW = 0$, would imply $\lambda_{BC} = 0$ as $dW = -dC = dC^*$. One can easily see that in general, with asymmetric steady states, we would have $\lambda_{BC} \neq 0$.

Intuitively, notice that the policymaker's FOC with respect to the net-wealth W_{t+1} would be the usual Euler equation, i.e. $\lambda_{BC,t} = E_t \beta \lambda_{BC,t+1} r_{1,t+1}$, and would have the same stochastic properties of the marginal utility of consumption.

Using the general efficiency condition (equation (18)) we can derive numerically the optimal portfolio.

We first consider the case of perfect spanning only two productivity shocks. Due to the absence of monetary-policy trade offs, PPI stability is the optimal policy, as discussed, among others, by Benigno and Benigno (2006) and by Devereux and Sutherland (2007b) in relation to portfolio choices. In this case we can compare the optimal portfolio allocation under PPI-stability obtained using the Devereux and Sutherland technique with the one obtained under the Ramsey policy using our algorithm. We find that the optimal holdings of domestic bonds, for our benchmark parameterization, are equal to $\alpha_0 = 2.29358$ under both, the Ramsey and the PPI-stability policies. This is reassuring as we know that the PPI-stability policy reproduces the first-best allocation which should be achieved by the Ramsey policymaker as well.

The second case refers to complete spanning but only two mark-up shocks. In this case PPI-stability is no longer the optimal monetary policy. Under this policy the optimal portfolio is again $\alpha_0 = 2.29358$ as mark-up shocks and productivity shocks have identical macroeconomic consequences.[48] Under the Ramsey optimal policy, instead, the optimal portfolio requires $\alpha_0 = 6.39368$. We can see that the optimal policy induces a larger holding of domestic bonds.

The final case we consider is that of incomplete spanning: all four shocks are present. In this case, the portfolio under PPI-stability is unchanged as having four shocks of this type amounts to taking a multiple of the variance of either of the pairs of shocks considered separately. Under the Ramsey policy, instead, the optimal portfolio requires $\alpha_0 = 2.56435$, which is closer to the one obtained with technology shocks.

Finally, it should be noticed that in all these cases the home agent goes long in bonds denominated in domestic currency. When we compute the optimal portfolio under a simple Taylor rule, we obtain the opposite result: i.e. agents go short in the bonds denominated in their own currency. We can see that both of the policies considered above, PPI-stability and Ramsey policy, make the return of the foreign assets comove positively with home GDP.

10 Conclusions

In this paper we have shown how to use standard perturbation methods to solve for asset market allocations in a general class of incomplete market economies with multiple agents and assets, in which portfolio choices are indeterminate

[48] The difference between these two shocks is that the technology shock shifts the aggregate disutility function of labor in a way that compensate the fluctuations in labor supply. The mark-up shocks generates fluctuations in the labor supply that are not compensated by shifts in the aggregate disutility of labor.

in the absence of uncertainty. This class of economies is more general than that analyzed by existing contributions such as Devereux and Sutherland (2008, 2007a) or Van Wincoop and Tille (2007), and is relevant in a number of interesting problems, for instance in solving for Ramsey optimal policies with multiple agents and assets under incomplete markets. Differently from Devereux and Sutherland, our general solution does not provide closed-form solutions but requires iterative methods, except in particularly simple cases.

We provide an application of our methods by solving for the optimal nominal bond portfolio under Ramsey monetary policy in a canonical 2-country economy with Calvo pricing and technology and mark-up shocks.

As a further contribution, we have also clarified the link between the Devereux-Sutherland solution methods and the asymptotic approach proposed by Judd and Guu 2001) to deal with bifurcations arising in static portfolio problems, showing that the two approaches rest on the same formal generalization of the Implicit Function Theorem provided by Bifurcation Theory.

Finally, we have shown how to use simple matrix algebra to extend the closed form solutions developed by Devereux and Sutherland (2008, 2007a) to solve for asset market equilibrium with more than two agents and, concerning their dynamics, also more than two assets, for the case in which portfolio allocations only appear multiplied by excess returns. Our extension is based on the fact that the optimal portfolio composition for each agent must be solved simultaneously with the portfolio of the other agents. Re-writing the (second-order accurate) state-space solution of the model in a particular matrix form, it is then possible to stack all agents' portfolio problem together, obtaining a simple linear system of equations. This is then solved with a standard matrix inversion.

Using our algebra it is straightforward to compute the optimal portfolio with any number of agents and assets. We show this by means of two applications widely discussed in the literature. The first consists of a three-country nominal-bond endowment economy. This application offers interesting insights on the portfolio composition that cannot be seen in a two-country setup. For example, assuming zero initial net-foreign-asset positions, an infinite variance of the monetary shock of one country would reduce all bond holdings to zero in the two-country model. In the three-country model, on the contrary, this infinite risk associated with one particular currency will only eliminate the bond holding in that currency for all countries.

The second application consists of a two-country model with trade in equities and bonds. This is a workhorse model for studying equity-home-bias issues. Solving for the optimal portfolio under complete markets (i.e. as many shocks as assets) we show that equity home bias is optimal only in a particular case: i.e. when all shocks are equally persistent. The relative persistence of the shocks, therefore, is an important determinant of the portfolio composition. Extending the model to an incomplete-market setup generates non-stationarity in the economic dynamics. By introducing a stochastic discount factor, and using our generalized portfolio solution, we can assess the effect of the unit-root in net-wealth on the portfolio composition. We show that the results are qualitatively unchanged.

Our generalized portfolio solution can be used to address a number of interesting questions in open and closed economy models with multiple agents and assets.

Appendix

The three-country model

The following list defines the notation used for the variables of the model. Table 1 reports the non-linear equations of the three-country model.[49]

List of Variables

Consumption	C
Money	M
Endowment	Y
Consumer Prices	P
Return on Bond	r
Nominal price of Bond	Z
Net Wealth	W

[49] A $(+1)$ denotes next period value of a variable. A (-1) denotes previous period value.

Table 1: The non-linear three-country model

Table 1: The non-linear three-country model

Asset Choice First Order Conditioins
$\left(C_a(+1)\right)^{-\rho}\left(r_a(+1)-r_b(+1)\right)=0$
$\left(C_c(+1)\right)^{-\rho}\left(r_c(+1)-r_b(+1)\right)=0$
Real Return on Bonds (definition)
$r_a=\frac{1}{P Z_a(-1)}$
$r_b=\frac{1}{P_b Z_b(-1)}$
$r_c=\frac{1}{P_c Z_c(-1)}$
Euler Equations (Pricing of Bonds)
$Z_a\left(C_a\right)^{-\rho}=\frac{\beta\left(C_a(+1)\right)^{-\rho}}{P(+1)}$
$Z_b\left(C_b\right)^{-\rho}=\frac{\beta\left(C_b(+1)\right)^{-\rho}}{P_b(+1)}$
$Z_c\left(C_c\right)^{-\rho}=\frac{\beta\left(C_c(+1)\right)^{-\rho}}{P_c(+1)}$
Resource and Budget Constraints
$C_a+C_b+C_c=e^{Y_c}+e^{Y_a}+e^{Y_b}$
$W_a=r_b W_a(-1)+e^{Y_a}-C_a+e^{\xi_a}+\alpha_{1,1}\left(r_a-r_b\right)+\alpha_{1,2}\left(r_c-r_b\right)$
$W_c=r_b W_c(-1)+e^{Y_c}-C_c+e^{\xi_c}+\alpha_{2,1}\left(r_a-r_b\right)+\alpha_{2,2}\left(r_c-r_b\right)$
Quantity Equations
$e^{M_a}=P e^{Y_a}$
$e^{M_b}=P_b e^{Y_b}$
$e^{M_c}=P_c e^{Y_c}$
Auxiliary Equations
$dr_a=r_a-r_b$
$dr_c=r_c-r_b$
$dC_a=C_a-C_b$
$dC_c=C_c-C_b$
Exogenous Shocks (variables in logs)
$Y_a=\zeta_{Y_a} Y_a(-1)+\varepsilon_{Y_a}$
$Y_b=\zeta_{Y_b} Y_b(-1)+\varepsilon_{Y_b}$
$Y_c=\zeta_{Y_c} Y_c(-1)+\varepsilon_{Y_c}$
$M_a=\zeta_{M_a} M_a(-1)+\varepsilon_{M_a}$
$M_b=\zeta_{M_b} M_b(-1)+\varepsilon_{M_b}$
$M_c=\zeta_{M_c} M_c(-1)+\varepsilon_{M_c}$
$\xi_a=\varepsilon_{\xi_a}$
$\xi_c=\varepsilon_{\xi_c}$
Continued on next page ...

The two-country model

For this model we use the same notation used for the three-country model, for the variables that are present in both models. The following lists the remaining variables. Table 2 reports the non-linear equations of the two-country model.[50]

Other Variables in the Two-Country Model

Dividend	D
Price of Equities	Z
Price of Bonds	Q
Return on Bond	rb
Return on Equities	re
Price of Domestic Goods	P_Y
Real Exchange Rate	RER
Demand of Country i for Goods of Country j	$C_{i,j}$
Stochastic discount factor	$\beta(C)$

[50] A $(+1)$ denotes next period value of a variable. A (-1) denotes previous period value.

Table 2: The non-linear two-country model

Table 2: The non-linear two-country model

Resource and Budget Constraint
$W_a = W_a(-1)\, rb_b + P_{Y,a}\,(Y_a + D_a) - C_a + e^{\xi} + \alpha\_1\,(rb_a - rb_b) + \alpha\_2\,(re_a - rb_b) + \alpha\_3\,(re_b - rb_b)$
$e^{Y_a} + e^{D_a} = C_{b,a} + C_{a,a}$
$e^{Y_b} + e^{D_b} = C_{a,b} + C_{b,b}$
Asset Choice First Order Condition
$(rb_a(+1) - rb_b(+1))\,(C_a(+1))^{-\rho} = 0$
$(re_b(+1) - rb_b(+1))\,(C_a(+1))^{-\rho} = 0$
$(re_a(+1) - rb_b(+1))\,(C_b(+1))^{-\rho} = 0$
Real Return on Bonds (definition)
$rb_a = \frac{1}{P_a Q_a(-1)}$
$rb_b = \frac{RER}{P_b Q_b(-1)}$
Euler Equations (Pricing of Bonds)
$(C_a)^{-\rho} = \frac{\beta(C_a)\beta\,(C_a(+1))^{-\rho}}{P_a(+1)Q_a}$
$\frac{(C_b)^{-\rho}}{RER} = \frac{\beta(C_b)\beta\,(C_b(+1))^{-\rho}}{P_b(+1)Q_b}$
Quantity Equations
$\frac{e^{M_a}}{P_a} = P_{Y,a}\left(e^{Y_a} + e^{D_a}\right)$
$\frac{e^{M_b}}{P_b} = \frac{P_{Y,b}\left(e^{Y_b} + e^{D_b}\right)}{RER}$
CES Aggregator and Demands
$C_a = \left(\mu^{\theta^{-1}}\,(C_{a,a})^{\frac{\theta-1}{\theta}} + (1-\mu)^{\theta^{-1}}\,(C_{a,b})^{\frac{\theta-1}{\theta}}\right)^{\frac{\theta}{\theta-1}}$
$C_b = \left(\mu^{\theta^{-1}}\,(C_{b,b})^{\frac{\theta-1}{\theta}} + (1-\mu)^{\theta^{-1}}\,(C_{b,a})^{\frac{\theta-1}{\theta}}\right)^{\frac{\theta}{\theta-1}}$
$1 = \left(\mu\,(P_{Y,a})^{1-\theta} + (1-\mu)\,(P_{Y,b})^{1-\theta}\right)^{(1-\theta)^{-1}}$
$C_{a,b} = (1-\mu)\,(P_{Y,b})^{-\theta}\,C_a$
$C_{b,a} = (1-\mu)\left(\frac{P_{Y,a}}{RER}\right)^{-\theta} C_b$
$C_{b,b} = \mu\left(\frac{P_{Y,b}}{RER}\right)^{-\theta} C_b$
Real Return on Assets (definition)
$re_a = \frac{P_{Y,a}e^{D_a} + Z_a}{Z_a(-1)}$
$re_b = \frac{P_{Y,b}e^{D_b} + Z_b}{Z_b(-1)}$
Auxiliary Equations
$rxb = rb_a - rb_b$
$rxe1 = re_a - rb_b$
$rxe2 = re_b - rb_b$
$cd = C_a - C_b - \frac{RER}{\rho}$
Continued on next page ...

Table 2 – Continued	
Stochastic Discount Factor	
$\beta\,(C_a) = (C_a)^{-\psi}$ $\beta\,(C_b) = (C_b)^{-\psi}$	
Exogenous Shocks (variables in logs)	
$D_a = \zeta_D\, D_a(-1) + \varepsilon_{D_a}$ $D_b = \zeta_D\, D_b(-1) + \varepsilon_{D_b}$ $Y_a = \zeta_Y\, Y_a(-1) + \varepsilon_{Y_a}$ $Y_b = \zeta_Y\, Y_b(-1) + \varepsilon_{Y_b}$ $M_a = \zeta_M\, M_a(-1) + \varepsilon_{M_a}$ $M_b = \zeta_M\, M_b(-1) + \varepsilon_{M_b}$ $\xi = \varepsilon_\xi$	
Continued on next page . . .	

Cross product notation

Notice that the solution equation of each variable has a term of the form (e.g. for equation 1)

$$z' A_1 \varepsilon = \varepsilon' A_1' z$$

This can be written as

$$(z' \otimes \varepsilon')\, vec\,(A_1') = vec\,(A_1')'\,(z \otimes \varepsilon)$$

Therefore, stacking each equation on top of the other would have

$$\underbrace{\begin{bmatrix} vec\,(A_1')' \\ \vdots \\ vec\,(A_n')' \end{bmatrix}}_{A} (z \otimes \varepsilon) = A vec\,(\varepsilon z')\,.$$

Other Kronecker rules

$$\epsilon \otimes r_x' = \varepsilon r_x'$$

$$vec\,(\varepsilon z')\, \varepsilon' = (z \otimes I_{m \times m})\, \varepsilon \varepsilon'$$

and also

$$\varepsilon vec\,(\varepsilon z')' = \varepsilon \varepsilon'\,(z \otimes I_{m \times m})'$$

Shift of endogenous state variables

The solution we are interested in is a function of the cross products of the state vector $z_t' = [x_{t-1},\, s_t]$, such that $E_t z_{t+1} = z_{t+1}$. Some solution algorithms would deliver a solution in terms of the state vector $\hat{z}_t' = [x_t,\, s_t]$. For example, as shown in Lombardo and Sutherland (2007) have

$$s_t = F_1 x_{t-1} + F_2 s_{t-1} + F_3 V_{t-1} + F_4 \Sigma \quad (57)$$

$$c_t = P_1 x_t + P_2 s_t + P_3 V_t + P_4 \Sigma \quad (58)$$

$$V_t = \tilde{\Phi} V_{t-1} + \tilde{\Gamma}\, \tilde{\varepsilon}_t + \tilde{\Psi} \tilde{\xi}_t \quad (59)$$

$$x_t = N\, x_{t-1} + \varepsilon_t \quad (60)$$

$$s_t^f = F_1 x_{t-1} + F_2 s_{t-1}^f \quad (61)$$

where $V_t = (\hat{z}_t \otimes \hat{z}_t)$ and $\tilde{\xi_t} = (\hat{z}_{t-1} \otimes \varepsilon_t)$, or

$$\hat{z}_t \equiv \begin{bmatrix} x_t \\ s_t \end{bmatrix} = \begin{bmatrix} N & 0 \\ F_1 & F_2 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ s_{t-1} \end{bmatrix} + \begin{bmatrix} 0 \\ F_3 \end{bmatrix} V_{t-1} + \begin{bmatrix} 0 \\ F_4 \end{bmatrix} \Sigma + \begin{bmatrix} I \\ 0 \end{bmatrix} \varepsilon_t \tag{62}$$

$$c_t = \begin{bmatrix} P_1 P_2 \end{bmatrix} \hat{z}_t + P_3 (\hat{z}_t \otimes \hat{z}_t) + P_4 \Sigma \tag{63}$$

$$(\hat{z}_t \otimes \hat{z}_t) = \tilde{\Phi} (\hat{z}_{t-1} \otimes \hat{z}_{t-1}) + \tilde{\Gamma} \tilde{\varepsilon}_t + \tilde{\Psi} (\hat{z}_{t-1} \otimes \varepsilon_t) \tag{64}$$

$$s_t^f = F_1 x_{t-1} + F_2 s_{t-1}^f \tag{65}$$

Say that we want to express c_t in terms of z_t. Then we should recognize that

$$\hat{z}_t = \underbrace{\begin{bmatrix} N & 0 \\ 0 & I \end{bmatrix}}_{U_1} z_t + \underbrace{\begin{bmatrix} I \\ 0 \end{bmatrix}}_{U_2} \varepsilon_t \tag{66}$$

Then replacing this in equation (63) we have

$$\begin{aligned} c_t &= \begin{bmatrix} P_1 P_2 \end{bmatrix} U_1 z_t + \begin{bmatrix} P_1 P_2 \end{bmatrix} U_2 \varepsilon & (67) \\ &+ P_3 ((U_1 z_t + U_2 \varepsilon) \otimes (U_1 z_t + U_2 \varepsilon)) + P_4 \Sigma & (68) \end{aligned}$$

Then, noting that for given matrices A, B, C and D we have

$$(A + B) \otimes (C + D) = (A \otimes C) + (A \otimes D) + (B \otimes C) + (B \otimes D)$$

we can rewrite

$$\begin{aligned} c_t &= \begin{bmatrix} P_1 P_2 \end{bmatrix} U_1 z_t + \begin{bmatrix} P_1 P_2 \end{bmatrix} U_2 \varepsilon + P_4 \Sigma & (69) \\ &+ P_3 ((U_1 z_t \otimes U_1 z_t) + (U_1 z_t \otimes U_2 \varepsilon)) \\ &+ P_3 ((U_2 \varepsilon \otimes U_1 z_t) + (U_2 \varepsilon \otimes U_2 \varepsilon)) \end{aligned}$$

or

$$\begin{aligned} c_t &= \begin{bmatrix} P_1 P_2 \end{bmatrix} U_1 z_t + \begin{bmatrix} P_1 P_2 \end{bmatrix} U_2 \varepsilon + P_4 \Sigma & (70) \\ &+ P_3 (U_1 \otimes U_1) (z_t \otimes z_t) + P_3 [(U_1 \otimes U_2) + (U_2 \otimes U_1) P_v] (z_t \otimes \varepsilon) \\ &+ P_3 (U_2 \otimes U_2) (\varepsilon_t \otimes \varepsilon_t) \end{aligned}$$

where P_v is a vector permutation matrix such that $P_v (z_t \otimes \varepsilon_t) = (\varepsilon_t \otimes z_t) = vec (z_t \varepsilon_t')$.

With reference to the portfolio solutions given in the text, notice that $D_5 = P_3 (i_C, :) [(U_1 \otimes U_2) + (U_2 \otimes U_1) P_v]$ where i_C indicates here the row corresponding to the consumption differential. Similarly $R_5 = P_3 (i_r, :) [(U_1 \otimes U_2) + (U_2 \otimes U_1) P_v]$ where i_r indexes the row corresponding to the excess return.

If the state-space solution was given in terms of $P_3 vech (\hat{z}_t \hat{z}_t')$, then can use the matrix L^h such that $L^h vech (\cdot) = vec (\cdot)$, to have $P_3 vech (\hat{z}_t \hat{z}_t') = P_3 L^h vec (\hat{z}_t \hat{z}_t')$.

Finally notice that the solution for the dynamics of the portfolio is given by

$$\alpha_t = \gamma' z_{t+1}$$
$$z_{t+1} = \begin{bmatrix} I & 0 \\ F1 & F2 \end{bmatrix} \hat{z}_t + \begin{bmatrix} 0 \\ F1 \end{bmatrix} \varepsilon_t \qquad (71)$$

References

Benigno, G. and Benigno, P. (2006). Designing Targeting Rules for International Monetary Policy Cooperation. *Journal of Monetary Economics*, 53:473–506.

Calvo, G. A. (1983). Staggered Prices in a Utility-Maximizing Framework. *Journal of Monetary Economics*, 12:383–98.

Clerc, L., Dellas, H., and Loisel, O. (2008). To Be or Not to be in monetary union. Technical report, University of Bern.

Coeurdacier, N., Kollmann, R., and Martin, P. (2007). International Portfolios with Supply, Demand and Redistributive Shocks. NBER Working Papers 13424, National Bureau of Economic Research, Inc.

Coeurdacier, N., Kollmann, R., and Martin, P. (2008). International Portfolios, Capital Accumulation and Foreign Asset Dynamics. Mimeo.

Dedola, L. and Straub, R. (2008). Optimized Reserve Holdings and Country Portfolios. Mimeo.

Devereux, M. and Sutherland, A. (2008). Country portfolios in open economy macro models. *NBER Working Paper*.

Devereux, M. B. and Sutherland, A. (2007a). Country Portfolio Dynamics. CEPR Discussion Papers 6208, C.E.P.R. Discussion Papers.

Devereux, M. B. and Sutherland, A. J. (2007b). Financial globalization and monetary policy. Mimeo.

Engel, C. and Matsumoto, A. (2006). Portfolio Choice in a Monetary Open-Economy DSGE Model. NBER Working Papers 12214, National Bureau of Economic Research, Inc.

Engel, C. and Matsumoto, A. (2008). International Risk Sharing: Through Equities or Bonds? Mimeo.

Evans, M. D. and Hnatkovska, V. (2007). Solving General Equilibrium Models with Incomplete Markets and Many Financial Assets. Manuscript, University of British Columbia.

Giavazzi, F. and Pagano, M. (1988). The advantage of tying one's hands EMS discipline and Central Bank credibility. *European Economic Review*, 32(5):1055–1075.

Heathcote, J. and Perri, F. (2007). The International Diversification Puzzle Is Not As Bad As You Think. NBER Working Papers 13483, National Bureau of Economic Research, Inc.

Jin, H.-H. and Judd, K. J. (2002). Perturbation methods for general dynamic stochastic models. Manuscript, Stanford University.

Judd, K. L. (1998). *Numerical Methods in Economics.* MIT Press, Cambridge, MA.

Judd, K. L. and Guu, S.-M. (2001). Asymptotic Methods for Asset Market Equilibrium Analysis. *Economic Theory*, 18:127–157.

Ljungqvist, L. and Sargent, T. J. (2000). *Recursive Macroeconomic Theory.* MIT Press, Cambridge, MA.

Lombardo, G. and Sutherland, A. (2007). Computing Second-Order-Accurate Solutions for Rational Expectation Models Using Linear Solution Methods. *Journal of Economic Dynamics and Control*, 31(2):515–530.

Lucas, R. (1982). Interest rates and currency prices in a two-country world. *Journal of Monetary Economics*, 10(3):335–359.

Neumeyer, P. (1998). Currencies and the Allocation of Risk: The Welfare Effects of Monetary Union. *American Economic Review*, 88(1):246–259.

Pavlova, A. and Rigobon, R. (2008). Equilibrium Portfolios and External Adjustment under Incomplete Markets. Manuscript, Sloan School of Management, MIT.

Ravenna, F. (2005). The european monetary union as a commitment device for new eu member states. Working Paper Series 516, European Central Bank.

Schmitt-Grohé, S. and Uribe, M. (2003). Closing Small Open Economy Models. *Journal of International Economics.* Forthcoming.

Schmitt-Grohé, S. and Uribe, M. (2004). Solving Dynamic General Equilibrium Models Using a Second-Order Approximation to the Policy Function. *Journal of Economic Dynamics and Control*, 28:645–858.

Sundaram, R. (1996). *A first course in optimization theory.* Cambridge University Press.

Van Wincoop, E. and Tille, C. (2007). International Capital Flows. NBER Working Papers 12856, National Bureau of Economic Research, Inc.

Figures

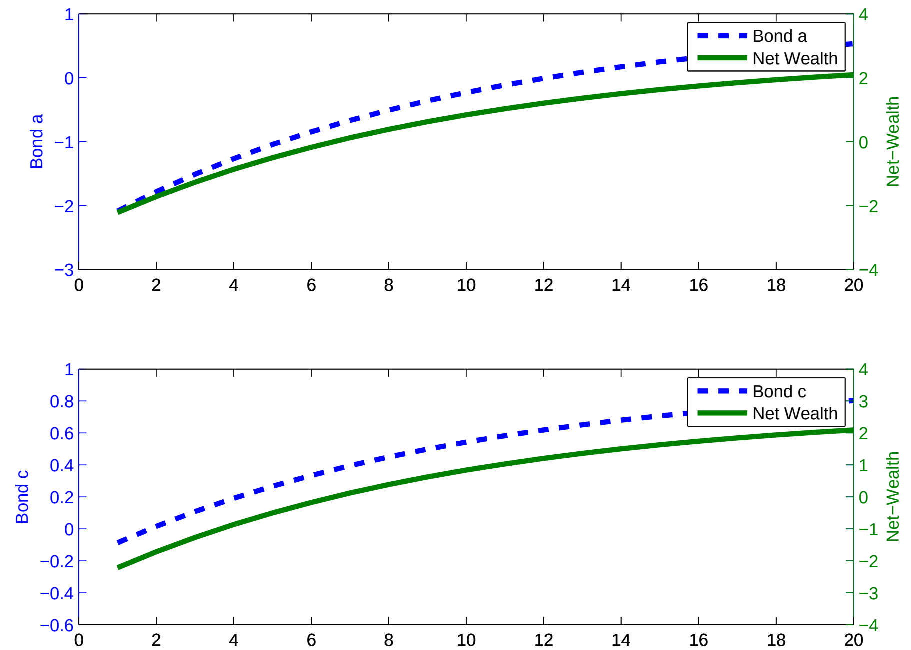

Figure 1: Response of country a's portfolio to country a's endowment shock

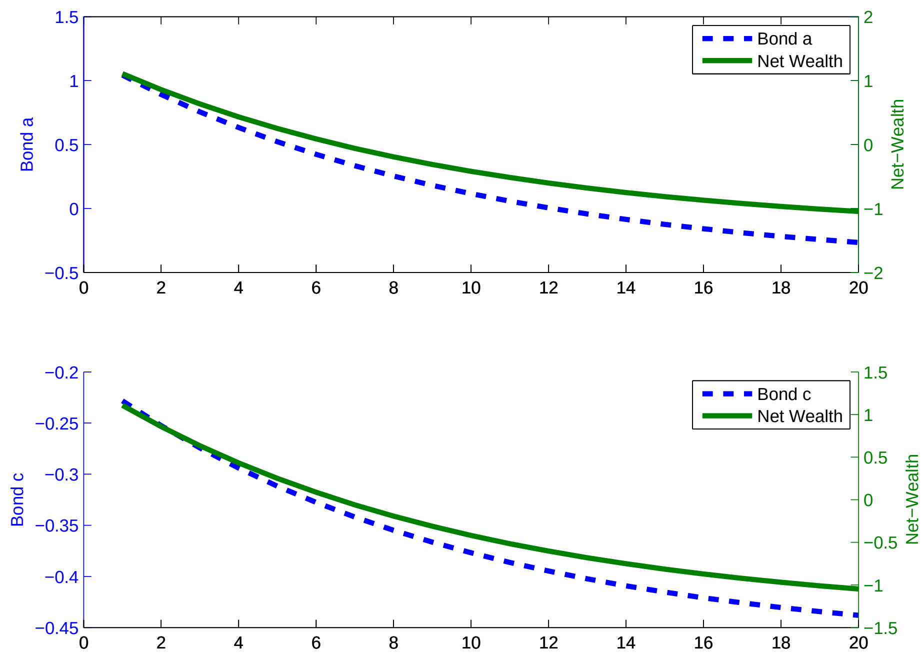

Figure 2: Response of country c's portfolio to country a's endowment shock

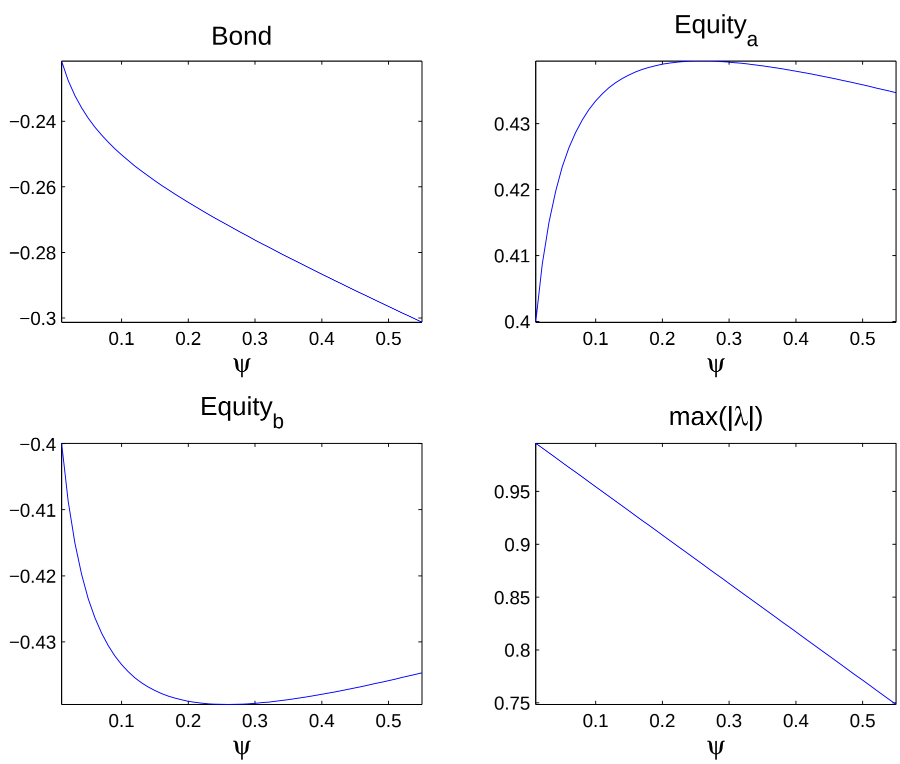

Figure 3: Sensitivity of steady-state portfolio to ψ (the vector λ contains the "stable" eigenvalues)

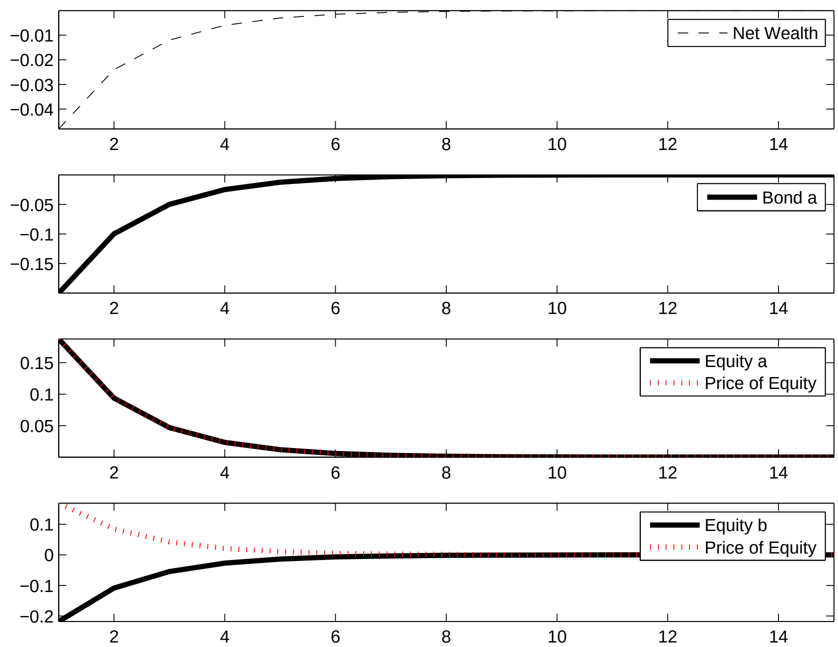

Figure 4: Response of country a's portfolio to a domestic dividend shock (complete markets)

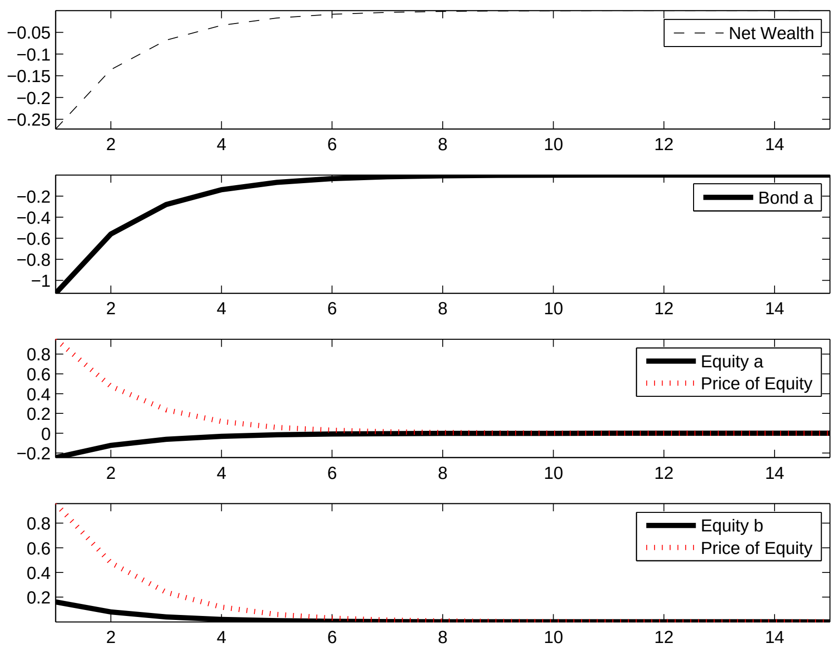

Figure 5: Response of country a's portfolio to a domestic endowment shock (complete markets)

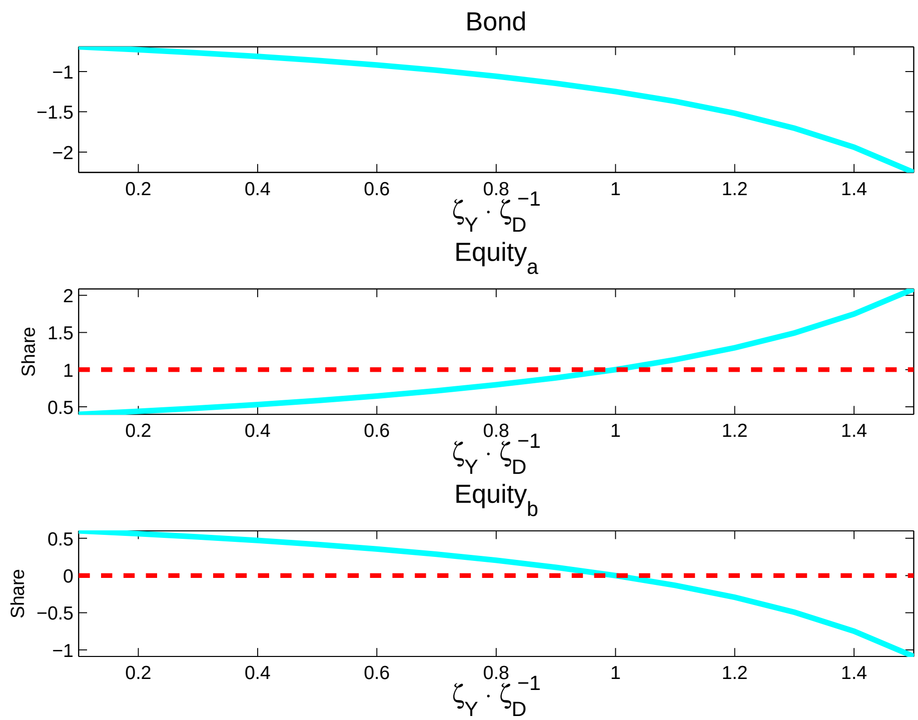

Figure 6: Equity home-bias and relative persistence of the shocks

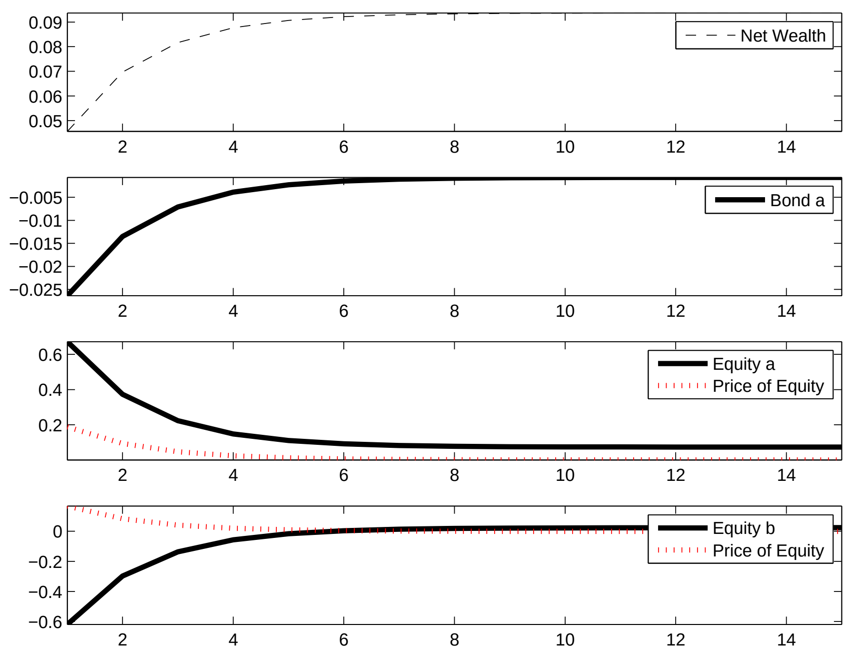

Figure 7: Response of country a's portfolio to a domestic dividend shock (incomplete markets)

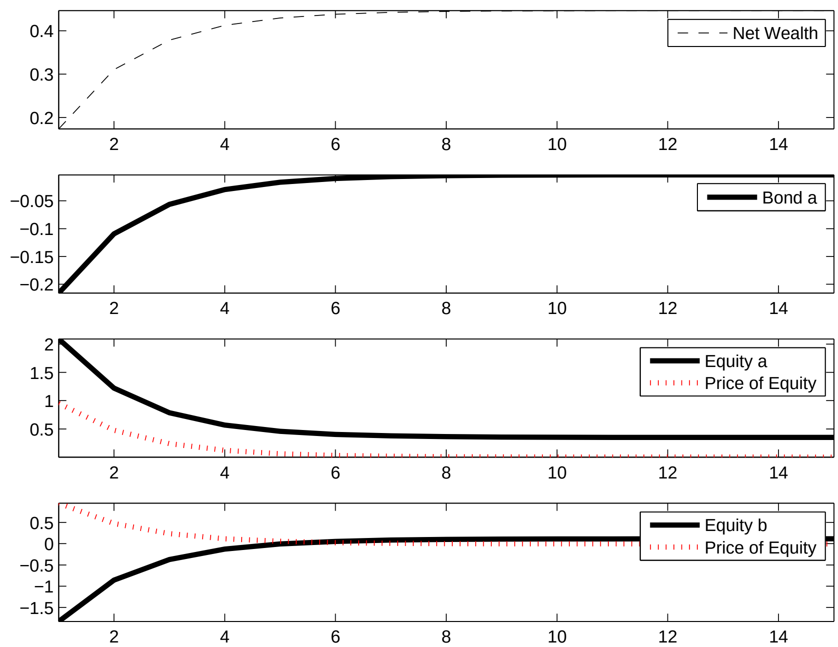

Figure 8: Response of country a's portfolio to a domestic endowment shock (incomplete markets)

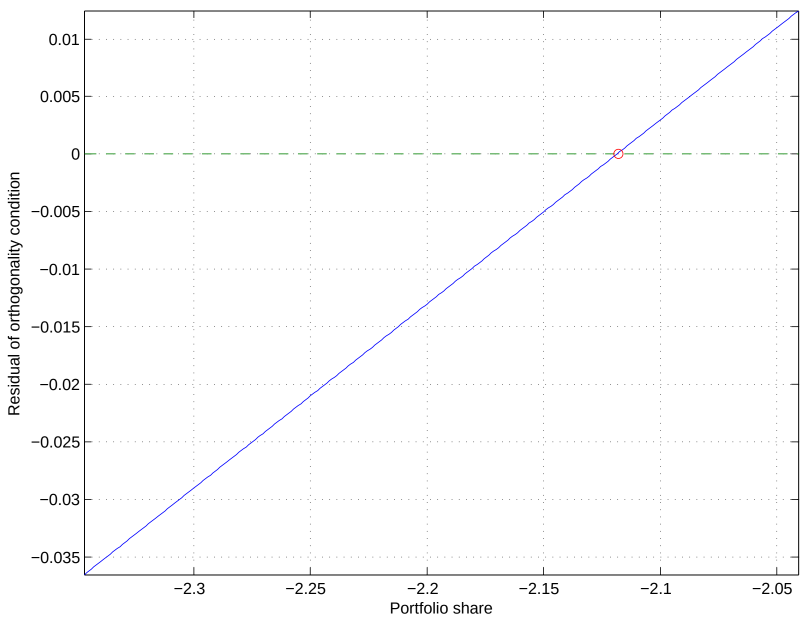

Figure 9: Residual of the orthogonality condition (7) and the optimal portfolio share