The First-Order Approach to Moral Hazard Problems with Hidden Saving

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Job Market Paper

November 2009

Abstract

Moral hazard models with hidden saving decisions are useful to study such diverse problems as unemployment insurance, income taxation, executive compensation, or human capital policies. How can we solve such models? In general, this is very difficult. Under the conditions derived in this paper, however, we can replace the incentive constraint with the associated first-order condition. This allows the application of simple Lagrangian methods and yields a precise characterization of optimal contracts. To obtain tractable conditions for the validity of this approach, the paper draws on the concept of log-convexity. Since log-convexity, unlike convexity, is preserved under multiplication, the paper is able to separate the assumptions on the output distribution from the assumptions on the agent's preferences in a sense, even though the interaction between these two is important for the agent's incentives.

The first-order approach is valid if the following conditions hold: a) the agent has non-increasing absolute risk aversion (NIARA) utility, b) the output technology has monotone likelihood ratios (MLR), and c) the distribution function of output is log-convex in effort (LCDF). Finally, the paper shows how the curvature of optimal wage schemes can be used to relax the above conditions.

Keywords: principal-agent problems, moral hazard, hidden savings, first-order approach, log-convexity

JEL Classification: C61, D82, D86, E21, H21

*Address: Department of Economics, University of Mannheim, 68131 Mannheim, Germany. Email: skoehne@mail.uni-mannheim.de. I would like to thank Ernst-Ludwig von Thadden, Nicola Pavoni, and Ian Jewitt for many constructive discussions. I received further helpful comments from Kaiji Chen, Marek Kapicka, Stefan Reichelstein, Richard Suen, Harald Uhlig, seminar participants in Mannheim, and conference participants at the Workshop on Public Economics 2009 Bonn, the German Economic Association (VfS) Annual Congress 2009 Magdeburg, and the European Winter Meeting of the Econometric Society 2009 Budapest.

1 Introduction

The study of moral hazard models is enormously simplified if one can use the first-order approach (Mirrlees 1974, Holmström 1979). By replacing the incentive constraint with the associated first-order condition, this approach allows the application of Lagrangian methods. The seminal works of Rogerson (1985) and Jewitt (1988) validate this procedure for the standard moral hazard problem. Very little is known, however, for more general environments. In particular, the validity of the first-order approach is not well understood for moral hazard problems in which the agent can secretly save (and borrow). This class of problems is particularly important, since observability of the agent's consumption-saving appears unrealistic for many common dynamic applications of the moral hazard framework (e.g., employment relationships, insurance problems, income taxation, etc.).

Under hidden saving, validating the first-order approach becomes significantly more complex. In addition to making sure that the agent's utility is at a global maximum with respect to the effort decision, one has to ensure the same for the saving decision, and most importantly for *joint* deviations to different effort and saving levels. Typically, the agent would combine a reduction of effort with an increased savings level to insure against the worsened output distribution. Therefore, ruling out joint deviations is the main difficulty in showing that first-order conditions imply incentive compatibility. The assumptions made by Rogerson (1985) and Jewitt (1988) are too weak for such a result, since they apply to the effort dimension only. In fact, given those assumptions, failure of the first-order approach under hidden saving is *typical* rather than exceptional (Kocherlakota 2004). Due to the two-dimensional decision space and the complementarity between shirking and saving, there is no obvious way of strengthening those assumptions, however.

The present paper validates the first-order approach for two-period moral hazard models with hidden saving. I show that the first-order approach is valid if the agent has nonincreasing absolute risk aversion (NIARA) utility, the output technology has monotone likelihood ratios (MLR), and the distribution function of output is log-convex in effort (LCDF). Note that a function is called *log-convex* if the logarithm of that function is convex. Any log-convex function is convex, but not vice versa. Hence, compared to Rogerson's (1985) assumption that the distribution function is convex in effort, LCDF is a stricter requirement. Under LCDF, the probability that output exceeds a given level is strongly concave in the agent's effort choice. Intuitively, this states that the marginal returns to effort are strongly decreasing in a particular sense.

The link from these conditions to the second-order effects of joint deviations is subtle. Note that by reducing his effort, the agent increases the probability of being punished by a low wage. By increasing his saving at the same time, he alleviates the severity of the punishment, since the utility difference between high and low wages will be reduced. Decreasing marginal returns to effort and a convex marginal utility of consumption limit the gains of such a strategy to some extent. The former implies that the probability of being punished increases more quickly than linearly as effort is reduced; the latter implies that the reduction of the punishment diminishes more quickly than linearly as saving is increased. Since the two effects are *multiplicative*, however, those properties are still too weak to ensure that a joint deviation is not attractive. At this point, the concept of log-convexity proves useful. In contrast to convexity, log-convexity is not only preserved under summation, but also under multiplication. Therefore, log-convexity of the agent's marginal utility of consumption (which is equivalent to NIARA) in combination with log-convexity of the distribution function (LCDF) implies that the 'punishment' described above is *jointly* log-convex (and hence convex) in effort and saving. This makes the agent's optimization problem jointly concave in his decision variables. Thus, first-order conditions yield incentive compatibility and the first-order approach is valid.

I also derive alternative sufficient conditions for the validity of the first-order approach. An important insight of the previous argument is the trade-off between convexity assumptions on the marginal utility of consumption on the one hand and convexity assumptions on the distribution function on the other hand. If one of the two functions is more convex than log-linear, then the assumption on the other function can be weakened. For a large class of utility functions, including CRRA utility, for example, this allows a relaxation of the LCDF condition. Finally, I show how to relax the LCDF condition by exploiting the curvature of the wage scheme. This allows me to validate the first-order approach for some interesting examples in which the LCDF property and even Rogerson's (1985) CDF condition fail. From a more general point of view, however, the curvature of the contract is not as helpful as in the standard moral hazard problem, since wages tend to become less concave in output under hidden saving.

The present paper is the first to identify the convexity properties of the distribution function and the agent's marginal utility of consumption as the key driving forces for the validity of the first-order approach under hidden saving. An equally important contribution is the introduction of log-convexity techniques. This idea gives optimization problems with multiplicatively separable objectives a tractable convex structure and seems to be useful for a much more general class of economic models.[1]

To the best of my knowledge, the only other result on the validity of the first-order approach under hidden saving is the work by Abraham and Pavoni (2009). However, they impose the 'spanning condition with dominance' from Grossman and Hart (1983), which is a severe restriction to the output technology. In fact, the condition imposes so much structure that moral hazard problems can be characterized quite well even without the first-order approach (Grossman and Hart 1983). Hence, assuming the spanning condition, it appears more natural to look for an extension of the method described in Grossman and Hart (1983), rather than for the validity of the first-order approach. The present paper does not need the spanning condition. The paper obtains the result by Abraham and Pavoni (2009) as a restrictive special case and goes much further.[2]

Establishing the validity of the first-order approach under hidden saving is valuable for a number of reasons. First, it yields a precise characterization of optimal contracts. Questions on the monotonicity of consumption or the value of information can be answered immediately, and one finds many analogies to the model without hidden saving. One also finds an important difference between the two models: Optimal wage schemes tend to be more convex in output under hidden saving. For a detailed discussion of these results, I refer the reader to the paper by Abraham, Koehne, and Pavoni (2009). In that paper, we also discuss implications for income taxation and conclude that optimal tax schemes are typically more 'regressive' compared to the setup with observable saving. Secondly, the first-order approach under hidden saving is helpful for a large range of applied questions. For instance, Bertola and Koeniger (2009) use this approach to develop a theoretical model on cross-country differences between public and private insurance. Gottardi and Pavoni (2009) build on the first-order approach to address optimal

[1] I am not aware of any other work that highlights this idea.

[2] Example 4 in Section 3 shows that the distribution function is log-convex in effort given the assumptions made by Abraham and Pavoni (2009). Moreover, Section 4 contains important relaxations of the log-convexity assumption.

capital taxation. Chade (2009) uses the first-order approach to study efficient compensation contracts. Finally, the first-order approach is important because it gives multi-period moral hazard problems with hidden saving a tractable recursive form, as discussed by Werning (2001), Werning (2002), Kocherlakota (2004), and others. Analytical results for the validity of the first-order approach provide a theoretical foundation for this procedure. The present paper marks an important first step towards this goal. However, the extension from the present two-period results to the multi-period problem remains a task for future research.[3]

The paper proceeds as follows: Section 2 describes the setup of the model. Section 3 validates the first-order approach given NIARA, MLR and LCDF. Section 4 shows how to relax the latter assumption. Section 5 collects all proofs. Section 6 concludes.

2 Setup

I study a two-period principal-agent problem. In the first period, the agent makes a hidden saving decision. In the second period, the agent exerts a hidden work effort. The contract is signed at the beginning of the first period and there is no renegotiation.

2.1 Preferences

The Principal (P) maximizes expected profits. P's discount factor is $1/R$, with $R > 0$. The Agent (A) has von-Neumann-Morgenstern preferences and maximizes the expected value of

$$u(c_1) + \beta \left(u(c_2) - v(e) \right),$$

where c_t denotes consumption and e represents effort. Consumption utility u is twice continuously differentiable and satisfies $u' > 0$, $u'' < 0$. Effort disutility v is twice continuously differentiable and satisfies $v' > 0$, $v'' \geq 0$.

[3]For the multi-period problem, it will be crucial to understand how the *value function* changes with the agent's savings level. A characterization of the value function is beyond the scope of the present paper, however. Note that conventional macroeconomic techniques are of limited help here, because the model involves a hidden state in combination with an *endogenously* determined probability distribution.

2.2 Technology

In the first period, A is endowed with w_0 units of the consumption good and can save at the rate $R > 0$. Negative saving, i.e., borrowing, is allowed. The set of feasible saving choices is the real interval J, which may be bounded or unbounded.[4] A's saving decision is not observable.

In the second period, A exerts an unobservable work effort $e \in I$, where I is a real interval. This generates a publicly observable stochastic output $x \in [\underline{x}, \overline{x}]$. (All results go through for discrete output spaces as well.) The output is distributed according to the probability density $f(x, e)$, which is continuously differentiable and has full support for all $e \in I$.

2.3 Contracts

At the beginning of the first period, P proposes a **contract** $(w(\cdot), e, s)$ consisting of an output-contingent wage scheme $w(\cdot)$ and recommended choices (e, s). A's utility from rejecting the contract is $\underline{U}$. The contract is called **optimal** if it maximizes expected profits subject to the incentive compatibility constraint and the participation constraint, i.e., if it solves the following problem:

$$\max_{w(\cdot),e,s} \frac{1}{R} \int_{\underline{x}}^{\overline{x}} (x - w(x)) f(x, e)\, dx \tag{P1}$$

s.t.

$$(e, s) \in \underset{(e',s') \in I \times J}{\operatorname{argmax}} \; u(w_0 - \frac{s'}{R}) + \beta \int_{\underline{x}}^{\overline{x}} u(w(x) + s') f(x, e')\, dx - \beta v(e') \tag{IC}$$

$$u(w_0 - \frac{s}{R}) + \beta \int_{\underline{x}}^{\overline{x}} u(w(x) + s) f(x, e)\, dx - \beta v(e) \geq \underline{U} \tag{PC}$$

2.4 First-order approach

Problem (P1) is extremely intricate. The incentive constraint (IC) consists of a two-dimensional continuum of inequalities. For all $e' \in I, s' \in J$, it requires

$$\begin{aligned} & u(w_0 - \frac{s}{R}) + \beta \int_{\underline{x}}^{\overline{x}} u(w(x) + s) f(x, e)\, dx - \beta v(e) \\ \geq\ & u(w_0 - \frac{s'}{R}) + \beta \int_{\underline{x}}^{\overline{x}} u(w(x) + s') f(x, e')\, dx - \beta v(e'). \end{aligned} \tag{1}$$

[4] The interval J may be bounded below due to a borrowing constraint and bounded above due to a nonnegativity constraint.

To obtain a problem that can be solved by standard methods, one replaces the incentive constraint by the agent's first-order necessary conditions. This gives rise to the following problem:

$$\max_{w(\cdot),e,s} \frac{1}{R}\int_{\underline{x}}^{\overline{x}} (x - w(x))f(x,e)\,dx \tag{P2}$$

s.t.

$$\beta \int_{\underline{x}}^{\overline{x}} u(w(x)+s)f_e(x,e)\,dx - \beta v'(e) = 0 \tag{FOCe}$$

$$\frac{1}{R}u'(w_0 - \frac{s}{R}) - \beta \int_{\underline{x}}^{\overline{x}} u'(w(x)+s)f(x,e)\,dx = 0 \tag{FOCs}$$

$$u(w_0 - \frac{s}{R}) + \beta \int_{\underline{x}}^{\overline{x}} u(w(x)+s)f(x,e)\,dx - \beta v(e) \geq \underline{U} \tag{PC}$$

Solutions to (P2) are denoted by $(w^*(\cdot), e^*, s^*)$. The associated consumption levels are denoted by $c_0^* = w_0 - s^*/R$ and $c^*(x) = w^*(x) + s^*$.

Replacing the true problem (P1) by the first-order problem (P2) is a valid procedure only if their solutions coincide. Assuming that the solutions to (P1) are interior with respect to effort and saving, this will be the case if and only if the contracts solving (P2) are incentive compatible. A sufficient condition for incentive compatibility is that the agent's decision problem is concave at those contracts. The remainder of this paper will identify conditions under which this is the case.

3 A sufficient condition for concavity of the agent's problem

In this section, I validate the first-order approach using nonincreasing absolute risk aversion, monotonicity of the wage scheme, and an assumption on the curvature of the output distribution function. This procedure strengthens the classic approach of Mirrlees (1979) and Rogerson (1985).

Using λ, μ and ξ as the Lagrange multipliers associated with the constraints (PC), (FOCe), (FOCs), respectively, the first-order condition of the Lagrangian of problem (P2) with respect to wages is

$$0 = -\frac{1}{R}f(x,e^*) + \mu\beta u'(c^*(x))f_e(x,e^*) - \xi\beta u''(c^*(x))f(x,e^*) + \lambda\beta u'(c^*(x))f(x,e^*), \quad x \in [\underline{x}, \overline{x}]. \tag{2}$$

Equivalently, as shown by Abraham and Pavoni (2009),

$$\frac{1}{R\beta u'(c^*(x))} = \lambda + \mu\frac{f_e(x,e^*)}{f(x,e^*)} + \xi\alpha(c^*(x)), \quad x \in [\underline{x}, \overline{x}], \tag{3}$$

where $\alpha(c) = -u''(c)/u'(c)$ is A's coefficient of absolute risk-aversion.

Expression (3) equates the principal's costs and benefits of marginally increasing the agent's utility at output x, normalized by the probability density. Compared to the standard moral hazard problem, there is now the additional term $\xi\alpha(c^*(x))$, because an increase of $u(c^*(x))$ relaxes the agent's Euler equation.[5]

I will often use the following two assumptions to give equation (3) more structure.

MLR. The likelihood ratio function, $f_e(x,e)/f(x,e)$, is continuously differentiable and nondecreasing in output x for all effort levels e.

NIARA. The agent's coefficient of absolute risk aversion, $\alpha(c) = -u''(c)/u'(c)$, is continuously differentiable and nonincreasing in consumption c.

MLR is standard and simply means that more output is indicative of higher effort. NIARA is also unproblematic, since it is satisfied by most common utility functions. NIARA implies that the multipliers λ, μ, ξ in the Kuhn-Tucker condition (3) are positive: $\lambda > 0$, $\mu > 0$, $\xi > 0$ (Abraham and Pavoni 2009). Moreover, MLR plus NIARA is sufficient for the wage scheme $w^*(x)$, with $w^*(x) = c^*(x) - s^*$, to be continuously differentiable and nondecreasing in output x; see equation (3).[6]

As noted before, the first-order approach is valid if A's objective function

$$(e,s) \mapsto u(w_0 - \frac{s}{R}) + \beta\int_{\underline{x}}^{\overline{x}} u(w^*(x)+s)f(x,e)\,dx - \beta v(e) \tag{4}$$

is concave in (e,s) for the wage scheme $w^*(\cdot)$ that solves (P2). One can restrict attention to A's second-period consumption utility as the next result shows.

[5]Note that an increase of $\beta u(c^*(x))$ by one marginal unit costs the principal $1/(R\beta u'(c^*(x)))$ units of consumption. On the other hand, it generates a benefit of λ because the participation constraint is relaxed and a benefit (or cost) of $\mu f_e/f$ because the first-order incentive constraint is relaxed (or tightened). In addition, there is a benefit of $\xi\alpha(c^*(x))$ because an increase of $\beta u(c^*(x))$ mitigates the agent's wish to save (Abraham, Koehne, and Pavoni 2009).

[6]NIARA can be relaxed. Equation (3) implies that $w^*(\cdot)$ is nondecreasing under MLR if $-(u'''u' - (u'')^2) \leq -u''(R\beta\xi)^{-1}$. This requires that the coefficient of absolute risk aversion does not increase too quickly.

Lemma 1. *A's decision problem is concave in* (e, s) *if A's second-period consumption utility*

$$(e, s) \mapsto \int_{\underline{x}}^{\overline{x}} u(w^*(x) + s) f(x, e)\, dx \tag{5}$$

is concave in (e, s).

By focusing on A's second-period consumption utility, I ignore the curvature generated by the effort disutility function v and by the effect of saving on first-period utility. In principle, one could obtain more general results by including these two effects. It not apparent, however, how far this would relax the curvature requirement imposed on the agent's second-period utility. Note, besides, that the role of the effort disutility function is limited anyway, since effort units can always be normalized such that this function is linear.

The following lemma identifies a sufficient condition for concavity of (5).

Lemma 2. *Suppose* $w^*(\cdot)$ *is continuously differentiable and nondecreasing. Suppose the distribution function of output,* $F(x, e) = \int_{\underline{x}}^{x} f(z, e)\, dz$, *is convex in* e *and for all* $x \in [\underline{x}, \overline{x}]$, $e \in I$, $s \in J$, *we have*

$$\frac{F_{ee}(x, e)F(x, e)}{(F_e(x, e))^2} \frac{u'''(w^*(x) + s)u'(w^*(x) + s)}{(u''(w^*(x) + s))^2} \geq 1. \tag{6}$$

Then A's second-period consumption utility is concave in (e, s).

To understand condition (6), note that $F_{ee}F/(F_e)^2$ is nonnegative if and only if F is convex in e, and at least 1 if and only F is log-convex in e.[7] Hence, $F_{ee}F/(F_e)^2$ measures the convexity of the distribution function F as a function effort. This motivates the following concept.

LCDF. The distribution function of output, $F(x, e) = \int_{\underline{x}}^{x} f(z, e)\, dz$, is log-convex in effort e for all output levels x.

A necessary but not sufficient condition for LCDF is that the distribution function is convex in effort. Hence, LCDF tightens the CDF condition from Mirrlees (1979) and Rogerson (1985). To interpret LCDF, note that $F(x', e)$ equals $1 - P(x > x'|e)$. Therefore, stating that $F(x', e)$ is log-convex in effort implies that the probability $P(x > x'|e)$ is 'strongly' concave in effort. For this reason, LCDF requires that the (stochastic) returns to effort are strongly decreasing:

[7] A function $f : \mathbb{R} \to \mathbb{R}_{++}$ is called *log-convex* if the logarithm of that function is convex. Assuming differentiability, f is log-convex if and only if $f''f/(f')^2 \geq 1$. Any log-convex function is convex, but not vice versa.

The probability $P(x > x'|e)$ that output is larger than some level x' is strongly concave in the agent's effort choice e for all values of x'.

Analogous to the interpretation of $F_{ee}F/(F_e)^2$, note that $u'''u'/(u'')^2$ is a measure of convexity of A's marginal utility of consumption. This measure is nonnegative if and only if u' is convex, and at least 1 if and only if u' is log-convex. Log-convexity of u' is equivalent to

$$\frac{u'''u' - (u'')^2}{(u')^2} \geq 0. \tag{7}$$

This is the case if and only if

$$\frac{d}{dc}\left(-\frac{u''(c)}{u'(c)}\right) \leq 0. \tag{8}$$

Hence, log-convexity of u' is equivalent to NIARA.

The main result is now a direct consequence of these observations: MLR, NIARA and LCDF validate the first-order approach.

Theorem 1. *Let* $(w^*(\cdot), e^*, s^*)$ *be a solution to the first-order problem* (P2)*. Suppose MLR, NIARA and LCDF. Then, given this contract, the agent's decision problem is concave. Hence, the contract solves the original problem* (P1)*.*

Compared to the model without hidden saving, Theorem 1 additionally requires NIARA and LCDF instead of Rogerson's (1985) CDF condition. NIARA is unproblematic, because it is satisfied by most common utility functions and confirmed by many empirical and experimental studies. As argued above, LCDF states that the returns to effort are strongly decreasing in a particular sense. The following examples, in particular the case with only two possible output levels, clarify this property.

Example 1 (Rogerson 1985)**.** Rogerson's paper contains the following distribution function that is convex in effort and satisfies MLR:

$$F(x, e) = \left(\frac{x}{\overline{x}}\right)^{e-\underline{e}}, \quad x \in [0, \overline{x}],\ e \in (\underline{e}, \infty). \tag{9}$$

This distribution function is not only convex in e, but even satisfies LCDF. Note

$$\log(F(x, e)) = (e - \underline{e}) \log\left(\frac{x}{\overline{x}}\right), \tag{10}$$

which shows that $F(x, e)$ is log-linear in e for all i.

Example 2 (Log-logistic distribution)**.** Let $0 < b \leq 1$. Consider the following distribution function:

$$F(x, e) = \frac{1}{1 + (e/x)^b}, \quad x \in [0, \infty), \; e \in (0, \infty). \tag{11}$$

It is not difficult to see that MLR is satisfied. Moreover, note

$$\log(F(x, e)) = -\log\left(1 + (e/x)^b\right). \tag{12}$$

Since $b \leq 1$, the expression $(e/x)^b$ is concave in e. Since the logarithm is increasing and concave, equation (12) shows that $\log(F(x, e))$ is convex in e. Thus, LCDF is satisfied.

The next two examples apply to discrete output spaces $X = \{x_1, \dots, x_n\}$, $x_i < x_j$ for $i < j$. In this setup, wages are vectors $(w_1, \dots, w_n) \in \mathbb{R}^n$, and probability weights $(p_1(e), \dots, p_n(e))$ replace the density function $f(x, e)$. The previous results extend to the discrete setup without difficulty.

Example 3 (Two outputs)**.** Consider the case with two possible outputs, $x_L < x_H$, and associated probabilities $p_L(e) = 1 - p(e)$, $p_H(e) = p(e)$, for some increasing function p with $0 \leq p(e) \leq 1$. Since p is increasing, MLR is satisfied. LCDF is equivalent to the log-convexity of $1 - p(e)$, which holds if and only if

$$\frac{-p''(e)(1 - p(e))}{(p'(e))^2} \geq 1 \text{ for all } e \in I. \tag{13}$$

To yield LCDF, the probability $p(e)$ of the high output level thus has to be sufficiently concave in effort. One example that satisfies this condition is the function $p(e) = 1 - \exp(-f(e))$, where $f : I \to (0, \infty)$ is increasing and concave.

Example 4 (Spanning condition)**.** Let $(\pi_{1h}, \dots, \pi_{nh})$, $(\pi_{1l}, \dots, \pi_{nl})$ be two probability distributions on $\{x_1, \dots, x_n\}$ such that π_{ih}/π_{il} is nondecreasing in i. (This implies that π_h first-order stochastically dominates π_l.) Let

$$p_i(e) = \Gamma(e)\pi_{ih} + (1 - \Gamma(e))\pi_{il} \tag{14}$$

for some increasing function Γ, with $0 \leq \Gamma(e) \leq 1$. Monotonicity of Γ, combined with the fact that π_{ih}/π_{il} is nondecreasing, yields MLR. Note

$$F_i(e) = F(x_i, e) = \sum_{j=1}^{i} p_j(e) = (1 - \Gamma(e)) \sum_{j=1}^{i} (\pi_{il} - \pi_{ih}) + \sum_{j=1}^{i} \pi_{ih}. \tag{15}$$

First-order stochastic dominance implies $\sum_{j=1}^{i}(\pi_{il} - \pi_{ih}) \geq 0$. Therefore, LCDF holds if $1 - \Gamma(e)$ is log-convex. This requirement is equivalent to

$$\frac{(\Gamma'(e))^2}{-\Gamma''(e)(1 - \Gamma(e))} \leq 1, \tag{16}$$

which is exactly the condition under which Abraham and Pavoni (2009) validate the first-order approach for the spanning condition and NIARA utility. Their proof relies heavily on the spanning condition and there is no obvious way how it might generalize to the setting considered in this paper. Moreover, Abraham and Pavoni's reading of the property in (16) is that the Frisch elasticity of leisure must not be larger than one (Abraham and Pavoni 2009, p. 16). This does not capture the true sense in which (16) tightens the CDF condition from Mirrlees (1979) and Rogerson (1985), in contrast to the argument provided in the present paper.

4 Alternative sufficient conditions for concavity

In this section, I discuss a few important relaxations of the assumptions made in Theorem 1. First, I study the case of CRRA utility functions. Then, I exploit curvature properties of the wage scheme. Note that, so far, the results have only used monotonicity of the wage scheme.

4.1 CRRA utility

Recall that the two crucial assumptions from Theorem 1, LCDF and NIARA, are strong convexity conditions for the distribution function and the agent's marginal utility of consumption, respectively. As Lemma 2 highlights, each of the conditions can be relaxed by strengthening the other. This insight is useful, because in many cases we do not only have log-convexity of the agent's marginal utility of consumption (NIARA), but stronger results.

The following proposition is a formal statement of this idea. It relaxes the LCDF property by restricting the class of preferences.

Proposition 3. *Let* $(w^*(\cdot), e^*, s^*)$ *be a solution to the first-order problem* (P2)*. Suppose MLR and NIARA. Suppose there exists a number* $\eta > 1$ *such that for all* c

$$\frac{u'''(c)u'(c)}{(u''(c))^2} \geq \eta, \tag{17}$$

and for all $e \in I$, $x \in [\underline{x}, \overline{x}]$,

$$\frac{F_{ee}(x,e)F(x,e)}{(F_e(x,e))^2} \geq \frac{1}{\eta}. \tag{18}$$

Then, given this contract, the agent's decision problem is concave. Hence, the contract solves the original problem (P1)*.*

Note that the right-hand side of (18) is a number between 0 and 1. Thus, the convexity requirement for the distribution function is somewhere between Rogerson's (1985) CDF condition and the LCDF property introduced in Theorem 1.[8]

As an important application of Proposition 3, consider CRRA utility: $u(c) = c^{1-\gamma}/(1-\gamma)$. Then we have

$$\frac{u'''(c)u'(c)}{(u''(c))^2} = 1 + \frac{1}{\gamma}. \tag{19}$$

Hence, using Proposition 3, we conclude that the first-order approach is valid if for all $e \in I$, $x \in [\underline{x}, \overline{x}]$,

$$\frac{F_{ee}(x,e)F(x,e)}{(F_e(x,e))^2} \geq \frac{\gamma}{1+\gamma}. \tag{20}$$

Under the spanning condition from Example 4, for instance, this property is equivalent to

$$\frac{(\Gamma'(e))^2}{-\Gamma''(e)(1-\Gamma(e))} \leq 1 + \frac{1}{\gamma} \quad \text{for all } e \in I. \tag{21}$$

This relaxes condition (16).

4.2 Exploiting the curvature of the contract

To validate the first-order approach, I have previously derived conditions on the output distribution and the agent's preferences under which, given that the contract is monotonic in output,

[8]Clearly, we cannot expect to obtain a better condition than CDF in the present setup, given the counterexample by Kocherlakota (2004).

the agent's decision problem is concave. Finding such conditions will be much simpler if the contract is not only monotonic, but also exhibits some form of concavity. For the moral hazard problem without hidden saving, Jewitt (1988) provides a general analysis of this idea. He shows that, under quite general conditions, the agent's utility changes with output in a concave way, and therefore the convexity conditions on the distribution function can be substantially relaxed.

In the present subsection, I try to exploit the curvature of the contract in the context of hidden saving.

4.2.1 Integration of the distribution function

For the standard moral hazard problem, Jewitt (1988, Theorem 1) validates the first-order approach when the agent's (ex-post) utility $u(w^*(x))$ is concave in output x and the integral of the distribution function satisfies a convexity property. He then relates the latter property to the concept of Total Positivity (Karlin 1968) and shows that it is satisfied for a rather general class of probability distributions. In the setup with hidden saving, such a result is more difficult to obtain. Since the agent's consumption level can be changed by saving, one needs to specify not only the curvature between utility and output, but also how this curvature changes with the savings level s. I obtain the following result.[9]

Proposition 4. *Let* $(w^*(\cdot), e^*, s^*)$ *be a solution to the first-order problem* (P2)*. Suppose MLR and NIARA. Suppose that for all output levels* x

$$-\frac{d^2(u(w^*(x)+s))}{dx^2} \quad \textit{is positive and log-convex in saving } s, \tag{C1}$$

$$\widetilde{F}(x,e) = \int_{\underline{x}}^{x} F(z,e)\,dz \quad \textit{is log-convex in effort } e. \tag{LCI}$$

Then, given this contract, the agent's decision problem is concave. Hence, the contract solves the original problem (P1)*.*

Unfortunately, there is no simple way of expressing condition (C1) in terms of the fundamentals of the model. However, it is easy to see that (C1) is a concavity property of the contract: A's (ex-post) consumption utility, $u(w^*(x)+s)$), depends on output x in a concave way. In

[9]To facilitate the argument, I suppose that the wage scheme is twice continuously differentiable. As the Kuhn-Tucker condition (3) shows, the wage scheme $w^*(x) = c^*(x) - s^*$ will be C^2 in x if $f_e(x,e)/f(x,e)$ is C^2 in x and $u'(c), \alpha(c)$ are C^2 in c.

addition, the curvature between utility and output changes with saving s in a log-convex way.

The next result shows that (C1) is satisfied if the wage scheme $w^*(x)$ is concave in output x. Hence, (C1) is guaranteed under an appropriate concavity property of the likelihood ratio function $f_e(x,e)/f(x,e)$; see the appendix for details.

Lemma 5. *Suppose NIARA and suppose $-u'''(c)/u''(c)$ is nonincreasing in c. Then condition* (C1) *is necessary but not sufficient for $w^*(x)$ to be concave in x.*

The assumption that $-u'''(c)/u''(c)$ is nonincreasing in c (nonincreasing absolute prudence) is innocuous. For instance, it is satisfied for all utility functions with hyperbolic absolute risk aversion (HARA).

To capture the second condition in Proposition 4, it is important to note that log-convexity is preserved under integration (Boyd and Vandenberghe 2004, p. 106). Therefore, log-convexity of the integral $\widetilde{F}(x,e) = \int_{\underline{x}}^{x} F(z,e)\,dz$ is a weaker assumption than log-convexity of $F(x,e)$ (LCDF). Intuitively, the integral $\widetilde{F}(x,e)$ will be log-convex in e if the distribution function $F(x,e)$ is log-convex in e for small values of x and "not too misbehaved" for large values of x. In fact, $F(x,e)$ does not even have to be convex in e as the following example shows.

Example 5 (Beta Prime distribution)**.** Consider the Beta Prime distribution with parameter $b = 2$:

$$f(x,e) = \frac{x^{e-1}(1+x)^{-e-2}}{B(e,2)}, \quad x \in [0,\infty),\ e \in (0,\infty), \tag{22}$$

where $B(e,b)$ represents the Beta function. The likelihood ratio function $f_e(x,e)/f(x,e)$ is nondecreasing concave in x, hence the class of preferences satisfying (C1) is nonempty. The distribution function is

$$F(x,e) = (1+e+x)x^e(1+x)^{-e-1}. \tag{23}$$

It is easy to see that $F(x,e)$ is not convex in e for all x. However, the primitive of the distribution function,

$$\widetilde{F}(x,e) = x\left(\frac{x}{1+x}\right)^e, \tag{24}$$

is log-linear in e. Therefore, LCI is satisfied.

Given this example, Proposition 4 even validates the first-order approach for a class of setups where the distribution function is not convex in effort. However, the gain of Proposition 4 in terms of relaxing the LCDF condition is limited. It is difficult to find many other examples that satisfy LCI without satisfying LCDF. In addition, condition (C1) is a stronger requirement than concavity of the agent's utility in output. Hence, compared to Jewitt's (1988) Theorem 1, it is much more difficult to relax the convexity properties of the distribution function in the present framework.

4.2.2 Quasiconvex distribution functions

To exploit the curvature of contracts, it is often helpful to study distribution functions that are (jointly) quasiconvex in output and effort, because, roughly speaking, such distributions are equivalent to production functions with nonincreasing returns to scale. For instance, Jewitt (1988, Theorem 3) and Conlon (2009) use this property to validate the first-order approach for multi-signal moral hazard problems. In the present section, I show that quasiconvex distribution functions also have attractive properties for the model with hidden saving.

Recall that, to establish concavity of the agent's decision problem, it is sufficient to consider the agent's utility in the second period (Lemma 1). Moreover, note that the distribution function $F(x, e)$ is quasiconvex in (x, e) if and only if output can be represented by a 'production function' $x = \varphi(e, z)$ that is nondecreasing concave in e and nondecreasing in the stochastic state of nature z (Jewitt 1988, Lemma 2). Using the production function, we can write the agent's second-period utility as

$$\int_{\underline{x}}^{\overline{x}} u(w^*(x) + s) f(x, e)\, dx = \mathbb{E}[u(w^*(\varphi(e, z)) + s)], \tag{25}$$

where $\mathbb{E}[\cdot]$ denotes expectations with respect to the state of nature z. Now, notice that concavity is preserved under summation and under nondecreasing concave transformations. Hence, since u is nondecreasing concave, the agent's decision problem will be concave in (e, b) if $w^*(\varphi(e, z))$ is concave in e for all z.[10] This insight is key for the following result.

Proposition 6. *Let* $(w^*(\cdot), e^*, s^*)$ *be a solution to the first-order problem* (P2). *Suppose that*

[10] At first glance, this insight seems to suggest that the validity of the first-order approach is not affected by the introduction of hidden saving. Notice, however, that the shape of the wage scheme is crucially influenced by the agent's ability to save; see equation (3).

the following conditions hold:

$$F(x,e) \text{ is quasiconvex in } (x,e), \tag{26}$$

$$f_e(x,e)/f(x,e) \text{ is nondecreasing and concave in } x \text{ for all } e, \tag{27}$$

$$g(c) := \left(\frac{1}{R\beta u'(c)} - \xi\alpha(c)\right) \text{ is increasing and convex in } c. \tag{28}$$

Then, given this contract, the agent's decision problem is concave. Hence, the contract solves the original problem (P1).

The function g defined in (28) links the likelihood ratio function $f_e(y,e)/f(y,e)$ to the shape of the wage scheme: Recall the Kuhn-Tucker condition (3),

$$\frac{1}{R\beta u'(c^*(x))} = \lambda + \mu\frac{f_e(x,e^*)}{f(x,e^*)} + \xi\alpha(c^*(x)), \quad x \in [\underline{x}, \overline{x}]. \tag{29}$$

Hence, the wage scheme $w^*(x) = c^*(x) - s^*$ is characterized by

$$w^*(x) = g^{-1}\left(\lambda + \mu\frac{f_e(x,e^*)}{f(x,e^*)}\right) - s^*, \quad x \in [\underline{x}, \overline{x}]. \tag{30}$$

Thus, since g^{-1} is nondecreasing concave under assumption (28), concave likelihood ratio functions will generate concave wage schemes $w^*(x)$ in this case.

As an important application of Proposition 6, consider CARA utility: $u(c) = -\exp(-\alpha c)/\alpha$. Then we have $g(c) = (R\beta)^{-1}\exp(\alpha c) - \xi\alpha$. Obviously, this function is increasing and convex in c. Therefore, Proposition 6 validates the first-order approach for CARA utility when the distribution function $F(x,e)$ is quasiconvex in (x,e) and the likelihood ratio function $f_e(x,e)/f(x,e)$ is nondecreasing concave in x.[11]

There are other examples, such as CRRA utility, for which g is not convex, however. In that case, the concavity property of the likelihood ratio function formulated in (27) has to be strengthened to obtain a concave wage scheme; the details can be found in the appendix. Essentially, the difficulty in obtaining concave wage schemes is driven by the coefficient of absolute risk aversion, which tends to make the right-hand side of the optimality condition (29)

[11] In the present paper, preferences over consumption and effort are additively separable. For CARA utility and *multiplicatively separable* preferences, by contrast, the validation of the first-order approach becomes much simpler, since the agent's effort choice will be independent of his wealth level. For most applications, this does not seem to be a useful approximation, however.

less concave in x. This result is also important for the characterization of optimal contracts and hints that wages become a more convex function of output under hidden saving.[12]

5 Proofs

Proof of Lemma 1. A's objective function is

$$(e, s) \mapsto u(w_0 - \frac{s}{R}) + \beta \int_{\underline{x}}^{\overline{x}} u(w^*(x) + s) f(x, e)\, dx - \beta v(e). \tag{31}$$

Since u is concave, the first summand is concave in (e, s). Since v is convex, the third summand is concave in (e, s). □

Proof of Lemma 2. Using partial integration, A's second-period consumption utility can be rewritten as

$$\int_{\underline{x}}^{\overline{x}} u(w^*(x) + s) f(x, e)\, dx = u(w^*(\overline{x}) + s) - \int_{\underline{x}}^{\overline{x}} (w^*)'(x) u'(w^*(x) + s) F(x, e)\, dx. \tag{32}$$

Hence, A's second-period consumption utility is concave in (e, s) if the function

$$(e, s) \mapsto - \int_{\underline{x}}^{\overline{x}} (w^*)'(x) u'(w^*(x) + s) F(x, e)\, dx \tag{33}$$

is concave, or equivalently if the function

$$(e, s) \mapsto \int_{\underline{x}}^{\overline{x}} (w^*)'(x) u'(w^*(x) + s) F(x, e)\, dx \tag{34}$$

is convex. I want to show that

$$g(e, s; x) = u'(w^*(x) + s) F(x, e) \tag{35}$$

is convex in (e, s) for all x. Since $(w^*)'(x) \geq 0$ by assumption, and since convexity is preserved under integration, this will imply convexity of (34).

The function $g(e, s; x)$ is convex in (e, s) if and only if its Hessian has a nonnegative diagonal

[12] See Abraham, Koehne, and Pavoni (2009) for a more detailed discussion of this insight.

and a nonnegative determinant. Omitting all arguments, the Hessian equals

$$H = \begin{pmatrix} F_{ee}u' & F_e u'' \\ F_e u'' & F u''' \end{pmatrix}. \tag{36}$$

The first diagonal entry is nonnegative by assumption. Condition (6) is equivalent to the statement that the determinant of H is nonnegative. In that case, the second diagonal entry of H must also be nonnegative. □

Proof of Theorem 1. By Lemma 1, it is sufficient to establish concavity of A's second-period consumption utility. Due to MLR and NIARA, the Kuhn-Tucker condition (3) implies that the wage scheme $w^*(x) = c^*(x) - s^*$ is continuously differentiable and nondecreasing in output x. Moreover, LCDF and NIARA imply that condition (6) from Lemma 2 is satisfied. Hence, A's second-period consumption utility is concave. □

Proof of Proposition 3. Using similar steps as in the proof of Theorem 1, the result follows from Lemma 1 and Lemma 2. □

Proof of Proposition 4. As Lemma 1 shows, it is sufficient to establish concavity of

$$(e, s) \mapsto \int_{\underline{x}}^{\overline{x}} u(w^*(x) + s) f(x, e)\, dx. \tag{37}$$

This is equivalent to establishing convexity of

$$(e, s) \mapsto -\int_{\underline{x}}^{\overline{x}} u(w^*(x) + s) f(x, e)\, dx. \tag{38}$$

Using two steps of partial integration, the latter function can be rewritten as

$$-u(w^*(\overline{x}) + s) + (w^*)'(\overline{x}) u'(w^*(\overline{x}) + s)) \widetilde{F}(\overline{x}, e) + \int_{\underline{x}}^{\overline{x}} \left(-\frac{d^2(u(w^*(x) + s))}{dx^2} \right) \widetilde{F}(x, e)\, dx. \tag{39}$$

First, note that the expression $-u(w^*(\overline{x}) + s)$ is convex in (e, s) due to the concavity of u. Moreover, the expression

$$(w^*)'(\overline{x}) u'(w^*(\overline{x}) + s)) \widetilde{F}(\overline{x}, e) \tag{40}$$

is convex in (e, s) by an argument similar to Lemma 2. For the third term in (39), note that

$$-\frac{d^2(u(w^*(x)+s))}{dx^2}\widetilde{F}(x,e) \tag{41}$$

is the product of a function that is log-convex in s and a function that is log-convex in e. Such products are convex in (e, s) as one easily verifies. Since convexity is preserved under integration, the third term in (39) is thus convex as well. This completes the proof. □

Proof of Lemma 5. Suppose $w^*(x)$ is concave in x. The function studied in (C1) can be represented as

$$-\frac{d^2(u(w^*(x)+s))}{dx^2} = -(w^*)''(x)u'(w^*(x)+s) + \left((w^*)'(x)\right)^2\left(-u''(w^*(x)+s)\right). \tag{42}$$

The first summand in (42) is log-convex in s, since $-(w^*)''(x) \geq 0$ and since u' is log-convex due to NIARA. The second summand is log-convex in s, since $((w^*)'(x))^2 \geq 0$ and since $-u''$ is log-convex when $-u'''/u''$ is nonincreasing. Since log-convexity is preserved under summation (Boyd and Vandenberghe 2004, p. 105), the function studied in (C1) is therefore log-convex in the variable s.

On the other hand, suppose that the function studied in (C1) is log-convex in s. As (42) shows, this does not imply that $(w^*)''(x)$ is nonpositive in general. □

Proof of Proposition 6. By Lemma 1, it is sufficient to consider A's second-period consumption utility. Moreover, due to quasiconvexity of the distribution function, the output technology can be represented by a production function $x = \varphi(e, z)$, with $\varphi(e, z)$ nondecreasing concave in effort e and nondecreasing in the stochastic state of nature z (Jewitt 1988, Lemma 2). Using this representation, we can write A's second-period consumption utility as

$$\int_{\underline{x}}^{\overline{x}} u(w^*(x)+s)f(x,e)\,dx = \mathbb{E}[u(w^*(\varphi(e,z))+s)], \tag{43}$$

where $\mathbb{E}[\cdot]$ denotes expectations with respect to the state of nature z.

I claim that w^* is nondecreasing concave. Recall from the Kuhn-Tucker condition (3) that solutions to (P2) are characterized by

$$w^*(x) = g^{-1}\left(\lambda + \mu \frac{f_e(x, e^*)}{f(x, e^*)}\right) - s^*, \tag{44}$$

with $g(c) = 1/(R\beta u'(c)) - \xi\alpha(c)$. By assumption, g is increasing and convex. Equivalently, g^{-1} is increasing and concave. Since $f_e(x, e^*)/f(x, e^*)$ is nondecreasing concave in x by assumption, this implies that $w^*(x)$ is nondecreasing concave in x.

Now, since $\varphi(e, z)$ is concave in e and $w^*(x)$ is nondecreasing concave in x, the composition $w^*(\varphi(e, z))$ is concave in e. Hence, the function $w^*(\varphi(e, z)) + s$ is concave in (e, s). Since u is nondecreasing concave, and since concavity is preserved under taking expectations, this completes the proof. □

6 Concluding remarks

This paper validates the first-order approach for two-period moral hazard problems with hidden saving. Compared to the model without hidden saving, I additionally impose an assumption on the convexity of the agent's marginal utility of consumption and a restriction of Rogerson's (1985) CDF condition. I obtain alternative sets of sufficient conditions by relaxing the latter property and including conditions on the curvature of the wage scheme.

These results show under what conditions the first-order approach can be safely applied. Besides, they indicate that the approach is slightly less general than in the standard moral hazard problem. Therefore, understanding how to characterize optimal contracts without applying the first-order approach could be a useful complement to the present work. Given the findings by Grossman and Hart (1983), however, such an approach will also involve strong structural assumptions in general.

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Appendix: Concave wage schemes

This section characterizes when the consumption scheme $c^*(x)$ solving the first-order problem (P2) is concave in output x. Since $w^*(x) = c^*(x) - s^*$, this property is equivalent to the wage scheme $w^*(x)$ being concave in x.

Due to equation (3), the consumption scheme is characterized by

$$c^*(x) = g^{-1}\left(\lambda + \mu L(x)\right), \tag{45}$$

with $g(c) = 1/(R\beta u'(c)) - \xi\alpha(c)$, $L(x) = f_e(x, e^*)/f(x, e^*)$. The first derivative of g equals

$$g'(c) = \frac{\xi u'''(c)u'(c) - ((R\beta)^{-1} + \xi u''(c))u''(c)}{u'(c)^2}, \tag{46}$$

which yields

$$(c^*)'(x) = \frac{\mu L'(x)u'(c^*(x))^2}{\xi u'''(c^*(x))u'(c^*(x)) - ((R\beta)^{-1} + \xi u''(c^*(x)))u''(c^*(x))}. \tag{47}$$

Omitting the arguments x and $c^*(x)$, the latter implies

$$\begin{aligned}(c^*)''(x) =& \frac{\mu}{(\dots)^2}\Big[\left(L''(u')^2 + 2L'u''u'(c^*)'\right)\left(\xi u'''u' - ((R\beta)^{-1} + \xi u'')u''\right) \\ &\qquad - L'(u')^2(c^*)'\left(\xi u^{(4)}u' - ((R\beta)^{-1} + \xi u'')u'''\right)\Big].\end{aligned} \tag{48}$$

Hence, given the assumption

$$(u')^2[\xi(u'''u' - (u'')^2) - u''(R\beta)^{-1}] > 0, \tag{49}$$

which is true under NIARA, $c^*(x)$ is concave in x if and only if the likelihood ratio function satisfies the following concavity condition:

$$\begin{aligned}L''(x) \le \frac{L'(c^*)'}{(u')^2[\xi(u'''u' - (u'')^2) - u''(R\beta)^{-1}]}&\Big[2\xi u'(-u'')(u'''u' - (u'')^2) + 2u'(u'')^2(R\beta)^{-1} \\ &+ \xi(u')^2(-u'')u''' + \xi(u')^3u^{(4)} - (u')^2u'''(R\beta)^{-1}\Big].\end{aligned} \tag{50}$$